The Minimal Robust Positively Invariant Set for Linear Difference Inclusions and its Robust Positively Invariant Approximations

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Abstract

Robust positively invariant (RPI) sets for linear difference inclusions are considered here under the assumption that the linear difference inclusion is absolutely asymptotically stable in the absence of additive state disturbances, which is the case for parametrically uncertain or switching linear discrete-time systems controlled by a stabilizing linear state feedback controller. The existence and uniqueness of the minimal RPI set and the minimal convex RPI set are studied. A new method for the computation of outer RPI approximations of the minimal RPI set for linear difference is presented; these approximations include a family of star–shaped RPI sets and two families of convex RPI sets. The use of a family of star–shaped RPI sets, and the characterization of the family, is reported for the first time.

Keyword: Set invariance, Minimal Robust Positively Invariant Set, Linear Difference Inclusions

1 Introduction

One of the fundamental tools employed in robust control of constrained dynamical systems is set invariance theory [1]. This is used in the design of reference governors [2] and predictive controllers [3–5] to guarantee constraint satisfaction, stability and convergence properties. One technique for robust control of constrained discrete-time systems is robust time-optimal control [6–9], which is based on the computation of a sequence of robust control (positively) invariant (RCPI) sets when the target set is also a RCPI set. A suitable target set in robust time-optimal control is the minimal robust positively invariant (mRPI) set [10]. The relevance of these ideas is demonstrated in the novel robust predictive controllers proposed recently in [11–13].

Computational issues and algorithmic procedures for the calculation of the (RCPI) sets and the application of these in robust control for constrained systems are discussed by a number of researchers [10, 14–22]. One of the outstanding problems for autonomous linear discrete-time systems is exact characterization of the mRPI set [1, 10, 23]. Several authors have developed procedures for the computation of outer approximations, with prespecified accuracy, of the mRPI set which are themselves RPI sets ; see, for instance, a procedure proposed in [24] and an alternative, simpler and improved, procedure in [25]. However, these papers address only approximation techniques for autonomous linear discrete-time invariant systems.

Linear difference inclusions (LDI) are used for modeling linear systems with parametric uncertainty; see, for example, [26, 27], and linear systems with switching dynamics [28].

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Uncertainty in the state transition matrix raises several issues, e.g. non–convexity of the forward reachable sets (as suggested by results reported in [26]). Such phenomena do not occur in the simpler case of autonomous linear discrete-time systems subject to bounded additive state uncertainty.

We establish that (i) under mild assumptions, the exact forward reachable sets are *star–shaped* sets [29, 30] , (ii) existence and uniqueness of the mRPI set and (iii) existence of three families of outer RPI approximations of the mRPI set. Non–convexity and computational complexity issues regarding the exact mRPI set are addressed by establishing existence and uniqueness of the minimal convex robust positively invariant (mCRPI) set. Additionally, we discuss computation of outer RPI approximations of the mCRPI set. Some technical and notational errors that appear in the conference version [31] of this work are corrected in this technical report. This technical report is based on the journal version of the paper [32].

Since the exact mRPI set is generally non–convex and it is very difficult to obtain a simple computational scheme for constructing it, three families of RPI sets are introduced. One consists of star–shaped RPI sets. This seems to be the first time that star–shaped RPI sets have been defined and studied. The other families consist of convex RPI sets: one contains convex sets that are outer–bounds for the mRPI set and the other contains the set that is tightest convex approximation of the MCRPI set as well as other convex sets.

Paper Structure

Section 2 is concerned with preliminaries. Existence and uniqness of the mRPI and the mCRPI sets are established in Section 3 . Section 4 contains a characterization of three families of RPI sets, a study of the limiting behaviour of increasingly accurate RPI approximations and a condition for characterization of a family of outer RPI approximations of the mCRPI set. Computational procedures for the case when the disturbance set is a polytope are given in Section 5 and some illustrative examples are in Section 6. Section 7 contains some conclusions. All proofs are given in the Appendix.

Basic Nomenclature and Definitions

Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}^+ \triangleq \{1, 2, \ldots\}$ and $\mathbb{N}_q \triangleq \{0, \ldots, q\}$, $\mathbb{N}_q^+ \triangleq \{1, \ldots, q\}$ for $q \in \mathbb{N}^+$. Further $\mathbb{B}_p^n(\gamma) \triangleq \{x \in \mathbb{R}^n | \; ||x||_p \leq \gamma\}$, where $|| \cdot ||_p$ denotes the vector p-norm, and $\mathbb{R}_+^n \triangleq \{x \in \mathbb{R}^n \mid x \geq 0\}$. Given an integer $s \in \mathbb{N}^+$ and sets $\Omega_i \subset \mathbb{R}^n$, the Minkowski set addition is defined by $\bigoplus_{i=1}^s \Omega_i \triangleq \Omega_1 \oplus \Omega_2 \oplus \ldots \oplus \Omega_s = \{\sum_{i=1}^s \omega_i \mid \omega_i \in \Omega_i\}$. For $\Omega \subset \mathbb{R}^n$, interior($\Omega$) and co($\Omega$) denote the interior and convex hull of Ω and $\lambda\Omega \triangleq \{\lambda\omega \mid \omega \in \Omega\}$ for any $\lambda \in \mathbb{R}$.

To clarify our use of the polyhedral/polytopic, C and star–shaped sets we recall the following definitions:

Definition 1 (Polyhedron/Polytope) *A* polyhedron *is the intersection of a finite number of open and/or closed half-spaces. A* polytope *is a closed and bounded polyhedron.*

Definition 2 (C set) *A set $\Omega \subset \mathbb{R}^n$ is a C set if it is a compact, convex set that contains the origin in its interior.*

Definition 3 (Star–Shaped set) *A set $\Omega \subset \mathbb{R}^n$ is a star–shaped set with a center at $\omega_c \in \mathbb{R}^n$ if $\omega_c \in \Omega$ and $\{\omega_c\} \oplus \lambda(\{-\omega_c\} \oplus \Omega) \subseteq \Omega$ for all $\lambda \in (0, 1]$. A star shaped set $\Omega \subset \mathbb{R}^n$ is basic if $\omega_c = 0$.*

We say that a set calculation is practicable when it can be carried out in finite time.

2 Preliminaries

We consider throughout the following linear difference inclusion:

$$\begin{aligned} x^+ &\in \mathcal{D}(x, \mathbb{A}, \mathbb{W}) \\ \mathcal{D}(x, \mathbb{A}, \mathbb{W}) &\triangleq \{Ax + w \mid A \in \text{co}(\mathbb{A}),\ w \in \mathbb{W}\} \\ \mathbb{A} &\triangleq \{A_i \in \mathbb{R}^{n \times n} \mid i \in \mathbb{N}_q^+\} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown disturbance and $q \in \mathbb{N}^+$ is a finite integer. The system is subject to an external additive state disturbance w that is contained in a C-set $\mathbb{W} \subset \mathbb{R}^n$. The system transition matrix A is uncertain and is known only to the extent that, at each time, it belongs to the convex hull of a finite set $\mathbb{A}$ of known and bounded matrices A_i so

$$A = \sum_{i=1}^{q} \lambda_i A_i,\ \ \lambda \in \Lambda \tag{2}$$

where λ can vary with time and

$$\Lambda \triangleq \{\lambda \in \mathbb{R}^q_+ \mid \sum_{i=1}^{q} \lambda_i \leq 1\} \tag{3}$$

We adopt the following standing assumption:

Assumption 1 *(i) The set $\mathbb{W}$ is a C-set in $\mathbb{R}^n$ and*
(ii) The matrix A at any time is given by (2) *for some (possibly time-varying) $\lambda \in \Lambda$.*

We refer to $\mathcal{D}(x, \mathbb{A}, \{0\})$ (i.e. when $\mathbb{W} = \{0\}$) as the nominal part of the linear difference inclusion (1).

The main motivation for considering linear difference inclusions of the form (1) lies in the fact that a broad class of systems can be modeled by this form. For example, consider the following uncertain, linear discrete-time system:

$$\begin{aligned} &x^+ = Fx + Gu + w\ ,\ \ (F, G) \in \mathrm{co}(\mathbb{C}),\ \ w \in \mathbb{W} \\ &\mathbb{C} \triangleq \{(F_i, G_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid i \in \mathbb{N}_q^+\} \end{aligned} \tag{4}$$

It is well known that system (4) with $\mathbb{W} = \{0\}$ is stabilized by $u = Kx$ if there exists a solution to the following, possibly conservative, linear matrix inequality problem [27]:

$$(F_i + G_iK)^T P(F_i + G_iK) - P < 0,\ \ P = P^T > 0,\ \ \forall i \in \mathbb{N}_q^+ \tag{5}$$

Motivated by (5) with $A_i \triangleq F_i + G_iK$, we assume that:

Assumption 2 *There exists a pair $(P, \psi) \in \mathbb{R}^{n\times n} \times (0, 1)$ such that $P = P^T > 0$ and*

$$A_i^T P A_i - P \leq -\psi P,\ \ \forall i \in \mathbb{N}_q^+ \tag{6}$$

Recalling results in [28, Section 3] on stability of linear difference inclusions when $\mathbb{W} = \{0\}$, it follows that if (6) holds then the linear difference inclusion (1) with $\mathbb{W} = \{0\}$ is *Absolutely Asymptotically Stable (AAS)* [28] so that $\lim_{k\to\infty} x(k) \to 0$, where $x(k)$ is generated by $x(i + 1) = A(i)x(i),\ x(0) = x_0$ for any $A(i) \in \mathbb{A}$ and any $x_0 \in \mathbb{R}^n$. It is also known that if (6) holds then the nominal part of the linear difference inclusions (1) (with $\mathbb{W} = \{0\}$) is AAS for all $A \in \mathrm{co}(\mathbb{A})$ [27, 33]. We also remark that the recent results reported in [34] can, in principle, be employed to relax Assumption 2, however we do not elaborate here on such a possibility.

Given a non-empty set $X \subset \mathbb{R}^n$, we use the following standard notation for the one step forward reachable set:

$$\mathcal{D}(X, \mathbb{A}, \mathbb{W}) \triangleq \{Ax + w \mid x \in X,\ A \in \mathrm{co}(\mathbb{A}),\ w \in \mathbb{W}\} \tag{7}$$

The following two definitions are standard definitions in set invariance theory (see [1, Section 2] and [10, Section 4]):

Definition 4 *A set Ω is a robust positively invariant (RPI) set of the difference inclusion* (1) *if $\mathcal{D}(\Omega, \mathbb{A}, \mathbb{W}) \subseteq \Omega$.*

Definition 5 *The set D^e_∞ is the minimal robust positively invariant (mRPI) set for the difference inclusion* (1) *over the class of closed RPI sets, if D^e_∞ is an RPI set and D^e_∞ is contained in every closed RPI set for the difference inclusion* (1).

Definition 6 *The set D_∞ is the minimal convex robust positively invariant (mCRPI) set for the difference inclusion* (1) *over class of closed RPI sets, if D_∞ is a convex RPI set and D_∞ is contained in every closed convex RPI set for the difference inclusion* (1)*.*

Clearly, from Definitions 5 and 6 it follows that, provided it exists, $D_\infty = \text{co}(D^e_\infty)$, because it is possible to show that $\text{co}(\Omega)$ is a RPI set for the difference inclusion (1) if Ω is such a set.

In order to discuss the convergence of the set sequences (taken in the Hausdorff metric sense) and to clarify our use of the term outer, RPI ε-approximation of the sets we recall the following two definitions:

Definition 7 *If Ω and Φ are two non-empty, compact sets in $\mathbb{R}^n$, then the Hausdorff metric is defined as*

$$\mathcal{H}(\Omega, \Phi) \triangleq \max\{\sup_{\omega\in\Phi} d(\omega, \Omega), \sup_{\phi\in\Omega} d(\phi, \Phi)\} \tag{8}$$

where $d(z, \mathcal{Z}) \triangleq \inf_{y\in\mathcal{Z}} ||z - y||_p$.

Definition 8 *Given a scalar $\varepsilon > 0$ and a non-empty set $\Omega \subset \mathbb{R}^n$, the set $\Phi \subset \mathbb{R}^n$ is an* outer ε-approximation *of Ω if $\Omega \subseteq \Phi \subseteq \Omega \oplus \mathbb{B}^n_p(\varepsilon)$ and an inner ε-approximation of Ω if $\Phi \subseteq \Omega \subseteq \Phi \oplus \mathbb{B}^n_p(\varepsilon)$.*

It is known that a collection of non-empty compact sets in $\mathbb{R}^n$, equipped with the Hausdorff Metric, forms a complete metric space [35]. A direct consequence is that every convergent or Cauchy sequence (whose elements belong to this collection) converges to an element of the space.

We also need the following definition in Section 5, where computational issues for the approximation of D_∞ are discussed:

Definition 9 *The support function $h_{\mathcal{X}}(\cdot)$ of a set $\mathcal{X} \subset \mathbb{R}^n$, evaluated at a vector $\eta \in \mathbb{R}^n$, is defined by:*

$$h_{\mathcal{X}}(\eta) \triangleq \sup_x \{\eta^T x \mid x \in \mathcal{X}\}.$$

Note that if $\mathcal{X}$ is a polytope then the *supremum* in Definition 9 is in fact *maximum*; furthermore, evaluation of $h_{\mathcal{X}}(\eta)$ is a linear programming problem.

For sake of convenience and compactness of the presentation in the sequel of this paper, we introduce the following additional notational convention. Let, for any $k \in \mathbb{N}^+$, $\mathbf{i}_k \triangleq \{i_k, i_{k-1}, \dots, i_2, i_1\}$ denote a sequence of integer variables such that $i_j \in \mathbb{N}^+_q$ for e ach $j \in \mathbb{N}^+_k$ and let for notational convenience $\mathbf{i}_0 \triangleq \{0\}$. We denote the set of all integer sequences $\mathbf{i}_k$ by $\mathcal{I}_k \triangleq \{\mathbf{i}_k \mid i_j \in \mathbb{N}^+_q,\ j \in \mathbb{N}^+_k\}$, $\forall k \in \mathbb{N}^+$ and $\mathcal{I}_0 \triangleq \{\mathbf{i}_0\}$. Given an arbitrary sequence $\mathbf{i}_k \in \mathcal{I}_k$ ($\mathbf{i}_k = \{i_k, i_{k-1}, \dots, i_2, i_1\}$) and an integer l such that $l \in \mathbb{N}_{k-1}$ we define $\mathbf{j}_{k-l}(\mathbf{i}_k) \triangleq \{i_k, i_{k-1}, \dots, i_{l+1}\}$ and $\mathbf{j}_0(\mathbf{i}_k) \triangleq \mathbf{i}_0$ so that $\mathbf{j}_{k-l}(\mathbf{i}_k) \in \mathcal{I}_{k-l}$ and for example, $\mathbf{j}_k(\mathbf{i}_k) = \mathbf{i}_k$, $\mathbf{j}_{k-1}(\mathbf{i}_k) = \{i_k, i_{k-1}, \dots, i_2\}$, ..., $\mathbf{j}_1(\mathbf{i}_k) = \{i_k\}$. We define the matrices $\mathcal{A}_{\mathbf{i}_k} \triangleq A_{i_k} \dots A_{i_1} A_{i_0}$ for arbitrary sequence $\mathbf{i}_k \in \mathcal{I}_k$ and $\mathcal{A}_{\mathbf{i}_0} \triangleq I$ where I is the identity matrix and $A_{i_j} \in \mathbb{A}$; this notational convention applies to matrices $\mathcal{A}_{\mathbf{j}_{k-l}(\mathbf{i}_k)}$ since $\mathbf{j}_{k-l}(\mathbf{i}_k) \in \mathcal{I}_{k-l}$.

3 The mRPI set D^e_∞ and the mCRPI set D_∞

In this section we discuss existence and uniqueness of the mRPI set D^e_∞ for the linear difference inclusion $\mathcal{D}(x, \mathbb{A}, \mathbb{W})$. Define the set sequence $\{D^e_k\}$ by:

$$D^e_{k+1} \triangleq \mathcal{D}(D^e_k, \mathbb{A}, \mathbb{W}),\ k \in \mathbb{N},\ D^e_0 = \{0\} \tag{9}$$

which describes the forward reachable tube [30, 36, 37] starting from the origin for the difference inclusion (1). A fundamental computational problem with respect to the set sequence $\{D^e_k\}$ is the fact that the sets $\{D^e_k\}$ are not necessarily convex. Given the exact set D^e_k of reachable states at time k, the convexity of D^e_{k+1} is not, generally, preserved due to multiplication of the uncertainty in the system transition matrix A and the state uncertainty. *This observation, discussed in [26] for the case $x^+ \in \mathcal{D}(X, \mathbb{A}, \{0\})$, is illustrated by a simple example in Section 6.*

3.1 Existence and Uniqueness of the mRPI set D^e_∞

It follows from (1) and (9) that each D^e_{k+1}, $k \in \mathbb{N}$, can be expressed as:

$$D^e_{k+1} = \bigcup_{x \in D^e_k} \mathcal{D}(x, \mathbb{A}, \mathbb{W}) = \bigcup_{x \in D^e_k} \{Ax + w \mid (A, w) \in \mathrm{co}(\mathbb{A}) \times \mathbb{W}\} \tag{10}$$

The exact calculation of the sets D^e_k by using (10) requires, clearly, uncountably many operations. Furthermore, the set D^e_{k+1} is possibly non–convex even if the set D^e_k is convex. These facts are also observed in [26] for a simpler case. Since $0 \in \mathrm{interior}(\mathbb{W})$ it follows that $0 \in \mathrm{interior}(D^e_k)$ for all $k \in \mathbb{N}^+$. Now, since $D^e_0 = \{0\} \subseteq \mathbb{W} = D^e_1$ and $\mathcal{D}(X, \mathbb{A}, \mathbb{W}) \subseteq \mathcal{D}(Y, \mathbb{A}, \mathbb{W})$ when $X \subseteq Y \subseteq \mathbb{R}^n$ it follows by induction that:

$$D^e_k \subseteq D^e_{k+1}, \ \forall k \in \mathbb{N} \tag{11}$$

In order to study the limiting behavior of the set sequence $\{D^e_k\}$, we introduce the sets $R^e_k, k \in \mathbb{N}$ defined by:

$$R^e_k \triangleq \mathcal{D}(R^e_{k-1}, \mathbb{A}, \{0\}), \ k \in \mathbb{N}^+, \ R^e_0 \triangleq \mathbb{W} \tag{12}$$

where $\mathcal{D}(X, \mathbb{A}, \{0\})$ is defined by (7). The set R^e_k is the set of states that can be reached at time k, *by the nominal part* $x^+ \in \mathcal{D}(x, \mathbb{A}, \{0\})$ *of the difference inclusion* (1), starting from an initial state that belongs to the set $\mathbb{W}$.

It is useful to know the structure of R^e_k, D^e_k before proceeding to study their limiting behaviour when $k \to \infty$; for this reason we establish next result:

Proposition 1 *Consider the sets* D^e_k *and* R^e_k, $k \in \mathbb{N}$, *defined by* (9) *(or* (10)*) and* (12). *Then:*

(i)

$$D^e_k \subseteq D^e_{k+1} \subseteq D^e_k \oplus R^e_k, \ \forall k \in \mathbb{N} \tag{13}$$

(ii) If Assumption 1 holds, the sets D^e_k *and* R^e_k, *are basic star–shaped sets.*

Proof: See appendix B1. Q.e.D.

By the proof of Proposition 1 it is clear that the above result hold when $\mathbb{W}$ is a basic star–shaped set.

Exploiting Proposition 1, we establish the following properties of the set sequence $\{D^e_k\}$:

Theorem 1 *Suppose Assumptions 1 and 2 hold. Then the set sequence* $\{D^e_k\}$ *defined by* (9) *satisfies :*
(i) There exist $\theta \in (0, 1)$ *and* $\mu < \infty$ *such that* $D^e_k \subseteq D^e_{k+1} \subseteq \mathcal{D}^e_k \oplus \theta^k \mathbb{B}^n_p(\mu)$ *for all* $k \in \mathbb{N}$ *,*
(ii) there exists a compact set D^e_∞ *such that* $\mathcal{H}(D^e_\infty, D^e_k) \to 0$ *as* $k \to \infty$*.*

Proof: See appendix B2. Q.e.D.

Theorem 1 establishes that $\{D^e_k\}$ is a Cauchy sequence of compact sets if Assumptions 1 and 2 hold. Since a family of compact sets equipped with the Hausdorff metric is a complete metric space, D^e_∞ is the unique limit of this Cauchy sequence and satisfies the following equation:

$$D^e_\infty = \mathcal{D}(D^e_\infty, \mathbb{A}, \mathbb{W}) \tag{14}$$

Consequently, robust positive invariance and minimality of the set D^e_∞ in (14) over the class of the closed RPI sets follows, since (i) $D^e_k \subseteq D^e_{k+1} \subseteq D^e_\infty$, $\forall k \in \mathbb{N}$, (ii) the sets D^e_k are not robust positively invariant and (iii) $\mathcal{D}(D^e_\infty, \mathbb{A}, \mathbb{W}) = D^e_\infty$. However, the main drawback is the fact that the set D^e_∞ does not generally admit an explicit representation.

3.2 The minimal convex RPI set D_∞

The fact that the set D^e_∞ exists and is unique for the class of closed RPI sets for the linear difference inclusion (1), implies directly existence and uniqueness of the minimal convex closed RPI set D_∞ for linear difference inclusion (1). However, since the sets D^e_k are non-convex, the set D^e_∞ is non–convex in general. Hence, it is generally difficult to use it in any practicable computational scheme. Furthermore, the use of the mCRPI set D_∞ is more appropriate for constrained control problems, when the constraints are convex. Since it is difficult to calculate D_∞ from D^e_∞ we resort to an alternative way to compute the set D_∞ and proceed to study the properties of the set sequence $\{D_k\}$ defined by:

$$D_{k+1} \triangleq \mathrm{co}\left(\bigcup_{j\in\mathbb{N}_q^+} A_j D_k\right) \oplus \mathbb{W},\ k\in\mathbb{N}, D_0=\{0\} \tag{15}$$

It follows from (15) that, for any finite integer $k\in\mathbb{N}^+$, the set D_k is a C set, since it is the Minkowski addition of a two C sets ($0\in\mathrm{interior}(\mathbb{W})$ it follows that $0\in\mathrm{interior}(D_k)$ for all $k\in\mathbb{N}^+$). Recalling, the set equality [38, Theorem 1.1.2]

$$\mathrm{co}(\mathcal{X}\oplus\mathcal{Y}) = \mathrm{co}(\mathcal{X})\oplus\mathrm{co}(\mathcal{Y}),\ \text{for all } \mathcal{X},\mathcal{Y}\subseteq\mathbb{R}^n, \tag{16}$$

the basic properties of Minkowski set addition and the fact that $\mathbb{W}$ is a C set, we have that:

$$D_{k+1} = \mathrm{co}\left(\left(\bigcup_{j\in\mathbb{N}_q^+} A_j D_k\right)\oplus\mathbb{W}\right) = \mathrm{co}\left(\bigcup_{j\in\mathbb{N}_q^+} (A_j D_k\oplus\mathbb{W})\right), \forall k\in\mathbb{N}^+ \tag{17}$$

An alternative description of the set sequence $\{D_k\}$, useful for the analysis in the sequel, is established by the next result.

Proposition 2 *Suppose Assumption 1 holds and consider the set sequence $\{D_k\}$ defined by* (15) *(or* (47)*). An equivalent form of that sequence is given by:*

$$D_{k+1} \triangleq \mathrm{co}\left(\bigcup_{\mathbf{i}_k\in\mathcal{I}_k} C_{\mathbf{i}_k}\right) \tag{18}$$

where the sets $C_{\mathbf{i}_k}$, $\mathbf{i}_k\in\mathcal{I}_k$, are defined by:

$$C_{\mathbf{i}_k} \triangleq \bigoplus_{l=0}^{k} \mathcal{A}_{\mathbf{j}_{k-l}(\mathbf{i}_k)}\mathbb{W} \tag{19}$$

or equivalently:

$$C_{\mathbf{i}_k} = \mathcal{A}_{\mathbf{j}_k(\mathbf{i}_k)}\mathbb{W}\oplus\mathcal{A}_{\mathbf{j}_{k-1}(\mathbf{i}_k)}\mathbb{W}\oplus\ldots\oplus\mathcal{A}_{\mathbf{j}_1(\mathbf{i}_k)}\mathbb{W}\oplus\mathcal{A}_{\mathbf{j}_0(\mathbf{i}_k)}\mathbb{W} \tag{20}$$

Proof: See appendix B3. Q.e.D.

Our next step is to exploit Proposition 2 in order to establish a relationship between the set sequences $\{D_k\}$ and $\{D^e_k\}$:

Proposition 3 *Suppose Assumption 1 holds and consider the set sequences $\{D^e_k\}$ and $\{D_k\}$ defined by* (9) *(or* (10)*) and* (15)*. Then*

$$D_k = \mathrm{co}(D^e_k), \forall k\in\mathbb{N}. \tag{21}$$

Proof: See appendix B4. Q.e.D.

Providing Assumptions 1 and 2 hold, by Lemma 2 in Appendix A and by Proposition 3 we have:

$$D_\infty = \mathrm{co}(D^e_\infty) \tag{22}$$

where D_∞ is the limit in the Hausdorff Metric of the set sequence $\{D_k\}$ (set equality holds since the limits exist). Since, in this case, the set sequence $\{D_k\}$ is Cauchy, D_∞ is its unique limit and consequently satisfies:

$$D_\infty = \mathrm{co}\left(\bigcup_{j\in\mathbb{N}_q^+} A_j D_\infty\right) \oplus \mathbb{W} \tag{23}$$

so that D_∞ is a convex RPI set. We have managed to describe D_∞ as the limit of a Cauchy set sequence. However, the main problem of obtaining an explicit description of D_∞ by means of practicable computation remains. It is difficult to obtain an explicit description of the set D_∞ even for the simple case when $q = 1$ (so that $\mathbb{A}$ is singleton) and the linear difference inclusion (1) is simply a linear time-invariant system subject to additive, bounded state disturbances and $D_\infty = D^e_\infty$, except maybe in some special cases [1, 10, 25]. It is, however, possible to obtain a family of outer RPI ϵ-approximations of the set D_∞ by exploiting ideas in [25] – such a family contains a practicably computable outer RPI ϵ-approximations of the set D_∞ as demonstrated in the sequel of this paper.

4 RPI approximations of the sets D^e_∞ and D_∞

Motivated by the fact that it is very difficult to obtain a simple computational scheme for constructing the sets D^e_∞ and D_∞, we provide in this section a description of three families of RPI sets. One consists of star–shaped RPI sets. This seems to be the first time that star–shaped RPI sets have been defined and studied. The other families consist of convex RPI sets: one contains convex sets that are outer–bounds for the mRPI set and the other contains the set that is tightest convex approximation of the MCRPI set as well as other convex sets. These families contain members that are practicably computable and consequently of practical relevance for control problems involving linear difference inclusions (1), for instance when considering robust control of linear difference inclusions subject to the state and control constraints. Star–shaped RPI sets are are of practical use when dealing with non–convex state constraints while the other two families are more suitable for the case of the convex constraints.

4.1 Family of Star–Shaped RPI Sets

A family of star–shaped RPI sets can be obtained by considering the set sequence $\{S_k\}$ defined by:

$$S_{k+1} \triangleq \bigoplus_{j=0}^{k} R^e_j,\ k\in\mathbb{N},\ S_0 \triangleq \{0\} \tag{24}$$

where the sets $R^e_k,\ k \in \mathbb{N}$, are defined in (12). When $\mathbb{W}$ is a C or basic star–shaped set with the origin in its interior, the sets S_k are star–shaped sets for any finite $k \in \mathbb{N}^+$, since they are the Minkowski sum of a finite number of star–shaped sets [29, Chapter 5, Section 5.3]; moreover $0 \in \mathrm{interior}(S_k)$ for all $k \in \mathbb{N}^+$, since $0 \in \mathrm{interior}(\mathbb{W})$. The sets S_k satisfy:

$$S_k \subseteq S_{k+1} = S_k \oplus R^e_k,\ \forall k \in \mathbb{N} \tag{25}$$

The relationship between the set sequences $\{D^e_k\}$ and $\{S_k\}$ is established next:

Proposition 4 *For the set sequences* $\{D^e_k\}$ *and* $\{S_k\}$*, defined by* (9) *and* (24)*:*

$$D^e_k \subseteq S_k, \forall k \in \mathbb{N}. \tag{26}$$

Proof: See appendix B5. Q.e.D.

If Assumptions 1 and 2 hold it is possible to demonstrate that $\{S_k\}$ is a Cauchy set sequence, by following the arguments and proof of Theorem 1; in which case the limit S_∞ of the set sequence $\{S_k\}$ exists, is unique and is given by:

$$S_\infty = \bigoplus_{j=0}^{\infty} R_j^e \tag{27}$$

The set S_∞ satisfies, by Proposition 4 and Theorem 1 and its analogue in Appendix A,:

$$D_\infty^e \subseteq S_\infty \tag{28}$$

and, by elementary properties of the linear difference inclusion (1) given in Appendix A, it follows that:

$$\begin{aligned} \mathcal{D}(S_\infty, \mathbb{A}, \mathbb{W}) &= \mathcal{D}(\bigoplus_{j=0}^{\infty} R_j^e, \mathbb{A}, \mathbb{W}) = \mathcal{D}(\bigoplus_{j=0}^{\infty} R_j^e \oplus \{0\}, \mathbb{A}, \mathbb{W}) = \mathcal{D}(\bigoplus_{j=0}^{\infty} R_j^e, \mathbb{A}, \{0\}) \oplus \mathcal{D}(\{0\}, \mathbb{A}, \mathbb{W}) \\ &\subseteq \bigoplus_{j=1}^{\infty} R_j^e \oplus \mathbb{W} = \bigoplus_{j=1}^{\infty} R_j^e \oplus R_0^e = \bigoplus_{j=0}^{\infty} R_j^e = S_\infty \end{aligned} \tag{29}$$

which is the desired robust positive invariance property of the set S_∞.

However, the set S_∞ is the Minkowski sum of a countably–infinite number of summands and is therefore of limited practical use. Additionally, it is generally very difficult to obtain a non–conservative estimate of the Hausdorff distance between the sets S_∞ and D_∞^e (or D_∞). Nevertheless, the main motivation for considering the set sequence $\{S_k\}$ is the fact that this sequence can be used to obtain practicably computable star–shaped RPI sets as demonstrated next.

We now introduce a condition that permits practicable calculation of RPI sets:

$$R_s^e \subseteq \alpha \mathbb{W} \tag{30}$$

where $(s, \alpha) \in \mathbb{N} \times [0, 1)$. A direct consequence of Assumptions 1 and 2 is the fact that $R_k^e \to \{0\}$ as $k \to \infty$. Hence there exist pair (s, α) satisfying $s < \infty$ and $\alpha \in (0, 1)$ such that (30) holds. Let:

$$\mathcal{P}_S \triangleq \{(s, \alpha) \in \mathbb{N} \times [0, 1) \mid R_s^e \subseteq \alpha \mathbb{W}\} \tag{31}$$

For $(s, \alpha) \in \mathcal{P}_S$, we define the following set:

$$S(s, \alpha) \triangleq (1 - \alpha)^{-1} \bigoplus_{j=0}^{s-1} R_j^e \tag{32}$$

We establish the robust positive invariance of the sets $S(s, \alpha)$ for arbitrary $(s, \alpha) \in \mathcal{P}_S$.

Theorem 2 *Suppose that Assumptions 1 and 2 hold. Then, the set $\mathcal{P}_S$ defined in* (31) *is non-empty. Moreover, given any pair $(s, \alpha) \in \mathcal{P}_S$ the set $S(s, \alpha)$ of* (32) *is a star–shaped RPI set for linear difference inclusion* (1) *and $D_\infty^e \subseteq S_\infty \subseteq S(s, \alpha)$.*

Proof: See appendix B6. Q.e.D.

Theorem 2 provides the description of a family $\mathcal{S}_S$ of star–shaped RPI sets defined by:

$$\mathcal{S}_S \triangleq \{S(s, \alpha) \mid (s, \alpha) \in \mathcal{P}_S\} \tag{33}$$

for which

$$\mathcal{D}(S(s,\alpha),\mathbb{A},\mathbb{W}) \subseteq S(s,\alpha),\ \forall S(s,\alpha) \in \mathcal{S}_S. \tag{34}$$

However, the main computational issues (such as, for example, complexity, explicit description and non–convexity) still remain. Since the computation of the sets R_k^e and hence S_k, is cumbersome, the results of Theorem 2 are theoretically interesting but of limited use. Motivated by this fact and the fact that most practical control problems are subject to convex state–control constraints we turn our attention to practicably computable convex RPI sets for linear difference inclusion (1).

4.2 Family of "Outer–Bounding" Convex RPI Sets

Next we study a family of "outer–bounding" convex RPI sets which have been considered in a conference paper [31]. Consider the set sequence $\{F_k\}$ defined by:

$$F_{k+1} \triangleq \bigoplus_{j=0}^{k} R_j,\ k \in \mathbb{N},\ F_0 \triangleq \{0\} \tag{35}$$

where the sets R_k are defined by:

$$R_k \triangleq \mathrm{co}\left(\bigcup_{\mathbf{i}_k \in \mathcal{I}_k} \mathcal{A}_{\mathbf{i}_k}\mathbb{W}\right),\ k \in \mathbb{N}^+,\ R_0 \triangleq \mathbb{W} \tag{36}$$

Note that, when $\mathbb{W}$ is a C set, the sets F_k are C sets for any finite $k \in \mathbb{N}^+$, since they are the Minkowski sum of a finite number of C sets ($0 \in \mathrm{interior}(F_k)$ for all $k \in \mathbb{N}^+$, since $0 \in \mathrm{interior}(\mathbb{W})$). We first establish the relationship between the set sequences $\{R_k^e\}$ and $\{R_k\}$:

Proposition 5 *Suppose Assumption 1 holds and consider the set sequences* $\{R_k^e\}$ *and* $\{R_k\}$ *defined by* (12) *and* (36) *respectively. Then*

$$R_k = \mathrm{co}(R_k^e),\ \forall k \in \mathbb{N}. \tag{37}$$

Proof: See appendix B7. Q.e.D.

By exploiting the results of Lemma 2 and Proposition 5, we can establish the following result:

Proposition 6 *Suppose Assumption 1 holds and consider the set sequences* $\{S_k\}$ *and* $\{F_k\}$ *defined by* (24) *and* (35) *respectively. Then*

$$F_k = \mathrm{co}(S_k),\ \forall k \in \mathbb{N}. \tag{38}$$

Proof: See appendix B8. Q.e.D.

If Assumptions 1 and 2 hold, by Lemma 2 and Proposition 6 we have (because the limits exist):

$$F_\infty = \mathrm{co}(S_\infty) \tag{39}$$

In this case (when Assumptions 1 and 2 hold), from Propositions 4 and 6 it follows that

$$D_k^e \subseteq S_k \subseteq F_k \tag{40}$$

for any $k \in \mathbb{N}$ as well as for the limit when $k \to \infty$, since the limits exist,:

$$D_\infty^e \subseteq S_\infty \subseteq F_\infty \tag{41}$$

When $\mathbb{W}$ is a C set (Assumption 1), the condition (30), by Proposition 5, is equivalent to:

$$R_s \subseteq \alpha\mathbb{W} \tag{42}$$

where $(s, \alpha) \in \mathbb{N} \times [0, 1)$ (because $R_s \subseteq \alpha\mathbb{W} \Rightarrow R_s^e \subseteq \alpha\mathbb{W}$ and $R_s^e \subseteq \alpha\mathbb{W} \Rightarrow R_s \subseteq \alpha\mathbb{W}$ since $\mathbb{W}$ is a C set and $R_s = \mathrm{R}_{\mathrm{s}}^{\mathrm{e}}$). We proceed as in the previous subsection and define:

$$\mathcal{P}_F \triangleq \{(s, \alpha) \in \mathbb{N} \times [0, 1) \mid R_s \subseteq \alpha\mathbb{W}\} \tag{43}$$

As in (32), we define

$$F(s, \alpha) \triangleq (1 - \alpha)^{-1} \bigoplus_{j=0}^{s-1} R_j, \ \forall (s, \alpha) \in \mathcal{P}_F \tag{44}$$

and establish the robust positive invariance of the sets $F(s, \alpha)$ for any pair $(s, \alpha) \in \mathcal{P}_F$.

Theorem 3 *Suppose that Assumptions 1 and 2 hold. Then, the set $\mathcal{P}_F$ defined in* (43) *is non-empty. Moreover, for any pair $(s, \alpha) \in \mathcal{P}_F$ the set $F(s, \alpha)$ of* (44) *is a C RPI set for linear difference inclusion* (1) *and $D_\infty^e \subseteq F_\infty \subseteq F(s, \alpha)$.*

Proof: See appendix B9. Q.e.D.

As in the previous subsection, Theorem 3 provides a family (set of sets) $\mathcal{S}_F$ of C RPI sets defined by:

$$\mathcal{S}_F \triangleq \{F(s, \alpha) \mid (s, \alpha) \in \mathcal{P}_F\} \tag{45}$$

Any set $F(s, \alpha) \in \mathcal{S}_F$ satisfies, by Theorem 3,

$$\mathcal{D}(F(s, \alpha), \mathbb{A}, \mathbb{W}) \subseteq F(s, \alpha) \tag{46}$$

From the computational point of view, the set sequence $\{F_k\}$ offers advantages since non–convexity is removed. Thus these results could be of practical use. However, it is in general difficult to obtain a non–conservative estimate of the Hausdorff distance between the sets F_∞ and D_∞^e (or D_∞) and, hence, between the sets $F(s, \alpha)$ and D_∞^e (or D_∞). We therefore do not explore further properties of the set sequences $\{S_k\}$ and $\{F_k\}$. Instead we proceed to study in more detail the set sequence $\{D_k\}$ defined by (15). However, we provide an RPI set $F(s, \alpha)$ and $S(s, \alpha)$ for an illustrative example in section 6.

4.3 Family of "Minimal" Convex RPI Sets

It is established in Sub–Section 3.2 that the set sequence $\{D_k\}$ of (6) satisfies $D_k = \mathrm{co}(D_k^e)$ for all $k \in \mathbb{N}$ as well as $D_\infty = \mathrm{co}(D_\infty^e)$ if Assumptions 1 and 2 hold. Since, in this case, the set D_∞ is the minimal convex RPI set we exploit the properties of the sets D_k to obtain a family of convex RPI sets that contains members that can be calculated in finite time. First, a method is established for the calculation of convex RPI sets, based on a simple , but appropriate, scaling of the sets D_k. Then, an additional condition is given that allows computation of convex RPI sets which are arbitrarily close approximations (using the Hausdorff metric) of $D_\infty = \mathrm{co}(D_\infty^e)$.

First, we recall that:

$$D_{k+1} \triangleq \mathrm{co}\left(\bigcup_{j \in \mathbb{N}_q^+} A_j D_k\right) \oplus \mathbb{W}, \ k \in \mathbb{N}, \ D_0 \triangleq \{0\} \tag{47}$$

It is worth mentioning that Proposition 2 implies directly that $D_k \subseteq F_k$ for any $k \in \mathbb{N}$ and that $D_\infty \subseteq F_\infty$ (providing Assumptions 1 and 2 hold) – in general D_∞ is a strict subset of F_∞ (an example in section 6 indicates $D_\infty \subset F_\infty$).

In order to obtain a characterization of practicably computable convex RPI sets, the following condition is required:

$$\mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)}\mathbb{W} \subseteq \alpha\mathbb{W},\ \forall \mathbf{i}_s \in \mathcal{I}_s \tag{48}$$

where, as before, $(s, \alpha) \in \mathbb{N} \times [0, 1)$. Since $\mathbb{W}$ is a C set, condition (48) is equivalent to:

$$\bigcup_{\mathbf{i}_s \in \mathcal{I}_s} \mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)}\mathbb{W} \subseteq \alpha\mathbb{W} \tag{49}$$

and consequently to:

$$\mathrm{co}\left(\bigcup_{\mathbf{i}_s \in \mathcal{I}_s} \mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)}\mathbb{W}\right) \subseteq \alpha\mathbb{W} \tag{50}$$

We note that, since $\mathbf{j}_s(\mathbf{i}_s) = \mathbf{i}_s$ by definition, conditions (42) and (48) are equivalent when $\mathbb{W}$ is convex; equivalence of conditions (30) and (42) is already established so consequently conditions (30), (42) and (48) are all equivalent if Assumption 1 holds.

Let:

$$\mathcal{P}_D \triangleq \{(s, \alpha) \in \mathbb{N} \times [0, 1) \mid \mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)}\mathbb{W} \subseteq \alpha\mathbb{W},\ \forall \mathbf{i}_s \in \mathcal{I}_s\} \tag{51}$$

Given any pair $(s, \alpha) \in \mathcal{P}_D$ we define the following set:

$$D(s, \alpha) \triangleq (1 - \alpha)^{-1} D_s \tag{52}$$

where D_s is given by (15) (or (47)). We now establish the robust positive invariance of the sets $D(s, \alpha)$:

Theorem 4 *Suppose that Assumptions 1 and 2 hold. Then, the set $\mathcal{P}_D$ defined in* (51) *is non-empty. Moreover, for any $(s, \alpha) \in \mathcal{P}_D$ the set $D(s, \alpha)$ of* (52) *is a C RPI set for linear difference inclusion* (1) *and $D^e_\infty \subseteq D_\infty \subseteq D(s, \alpha)$.*

Proof: See appendix B10. Q.e.D.

Theorem 4 provides a description of the following family $\mathcal{S}_D$ of "minimal" C RPI sets:

$$\mathcal{S}_D \triangleq \{D(s, \alpha) \mid (s, \alpha) \in \mathcal{P}_D\} \tag{53}$$

Theorem 4 can be used to develop and implement an algorithm for the computation of convex RPI approximation of D_∞. Clearly, from Theorem 4, the set $D(s, \alpha)$ is an outer RPI approximation of D_∞. However, the former can be a poor approximation of the latter; hence we proceed to present an extension of the results for the LTI systems case, which were reported in [25], in order to provide a way to obtain a set $D(s, \alpha)$ which is an approximation of pre–specified precision to D_∞ in that $D_\infty \subseteq D(s, \alpha) \subseteq D_\infty \oplus \mathbb{B}^n_p(\varepsilon)$ for an *a-priori* given $\varepsilon > 0$.

4.3.1 Limiting Behavior of the RPI set $D(s, \alpha)$

In order to be able to evaluate the quality of the approximation, in the Hausdorff metric sense, we have to study the limiting behaviour of $D(s, \alpha)$ with respect to the increase of s and the decrease of α i.e. how well $D(s, \alpha)$ approximates D_∞ if we choose sufficiently large s or a sufficiently small α. Given any $\alpha \in [0, 1)$, the smallest value of s such that (48) holds is:

$$s^0(\alpha) \triangleq \inf_s\{s \in \mathbb{N}^+ \mid R_s \subseteq \alpha\mathbb{W}\} \tag{54}$$

The smallest α such that (48) holds for a given $s \in \mathbb{N}^+$ is:

$$\alpha^0(s) \triangleq \inf_\alpha\{\alpha \in \mathbb{R}_+ \mid R_s \subseteq \alpha\mathbb{W}\} \tag{55}$$

Note that, for any $\alpha \in (0,1)$ the value of $s^0(\alpha)$ in (54) is finite and that $\alpha^0(s) \in [0,1)$ if and only if s is sufficiently large.

The following two theorems extend the results established in [25] for linear systems to the class of linear difference inclusions (1).

The first theorem addresses the issue of the limiting behaviour of $D(s,\alpha)$:

Theorem 5 *Suppose Assumptions 1and 2 hold, then*

i) $D(s,\alpha^0(s)) \to D_\infty$ *as* $s \to \infty$

ii) $D(s^0(\alpha),\alpha) \to D_\infty$ *as* $\alpha \searrow 0$

Proof: See Appendix B11. Q.e.D.

Theorem 5 implies that $D(s,\alpha)$ converges to D_∞ as $s \to \infty$ or $\alpha \searrow 0$. Thus, by increasing s and calculating α from (55), or by decreasing α and calculating s from (54), one can obtain a better approximation of D_∞. However, given a pre-specified accuracy, it is not clear yet how to obtain a pair (s,α) such that $D(s,\alpha)$ efficiently approximates D_∞ with the given accuracy.

This issue is dealt in with the next theorem, which provides conditions on (s,α) which guarantee that $D(s,\alpha)$ is an outer RPI ε-approximation of the mCRPI set D_∞.

Theorem 6 *Suppose Assumptions 1 and 2 hold, then for all* $\varepsilon > 0$ *there exists a pair* $(s,\alpha) \in [0,1) \times \mathbb{N}^+$ *such that* (48) *and*

$$\alpha(1-\alpha)^{-1}D_s \subseteq \mathbb{B}_p^n(\varepsilon) \tag{56}$$

hold. Moreover, for any pair $(s,\alpha) \in [0,1) \times \mathbb{N}^+$ *such that* (48) *and* (56) *hold, the set* $D(s,\alpha)$ *is an outer RPI* ε-approximation *of* D_∞.

Proof: See Appendix B12. Q.e.D.

Theorem 6 clearly states that given an *a priori* $\varepsilon > 0$, a collection of (s,α) can be found to satisfy (48) and (56). Then, any set $D(s,\alpha)$ is an outer RPI ε-approximation of D_∞, i.e. $D_\infty \subseteq D(s,\alpha) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$. Given $\varepsilon > 0$, let:

$$\mathcal{P}_{(D,\varepsilon)} \triangleq \{(s,\alpha) \in \mathbb{N} \times [0,1) \mid R_s \subseteq \alpha\mathbb{W},\ \alpha(1-\alpha)^{-1}D_s \subseteq \mathbb{B}_p^n(\varepsilon)\} \tag{57}$$

Then, by Theorem 6, the family of sets $\mathcal{S}_{(D,\varepsilon)} \subseteq \mathcal{S}_D$ defined by:

$$\mathcal{S}_{(D,\varepsilon)} \triangleq \{D(s,\alpha) \mid (s,\alpha) \in \mathcal{P}_{(D,\varepsilon)}\} \tag{58}$$

is a family of convex outer RPI ε-approximations of D_∞.

Let $M(s) \triangleq \sup_z\{\|z\|_p \mid z \in D_s\}$ and $M_\infty \triangleq \sup_z\{\|z\|_p \mid z \in D_\infty\}$ and note that these values are attained since D_s and D_∞ are compact sets. Since $D_s \subseteq D_\infty,\ \forall s \in \mathbb{N}$ it follows that $M(s) \le M_\infty$ and

$$\alpha \le \varepsilon(\varepsilon + M_\infty)^{-1} \le \varepsilon(\varepsilon + M(s))^{-1} \tag{59}$$

Hence, an upper bound for α can be obtained by using (59). Note also that (48) gives a lower bound for α such that $D(s,\alpha)$ is a RPI set that contains D_∞.

Remark 1 *The results of this subsection can easily be extended to the RPI sets* $F(s,\alpha)$ *and* $S(s,\alpha)$ *considered above; this extension is straight–forward and is therefore omitted.*

5 Computational Issues

We consider in more detail computational issues regarding the computation of an RPI set $D(s,\alpha)$. *However, similar computational schemes can be employed for computation of the sets $F(s,\alpha)$.* An algorithm for the computation of an RPI set $D(s,\alpha)$ satisfying $D_\infty \subseteq D(s,\alpha) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$ for a given $\varepsilon > 0$ can be formulated from Theorems 4 and 6 by observing that the lower and upper bounds imposed on α are specified by (48) (or equivalently by either (30) or (42) since $\mathbb{W}$ is a polytope)and (59) respectively. When $\mathbb{W}$ is a polytope, the pair (s,α) and $M(s)$ can be calculated *without having to compute explicitly* any of the afore-mentioned sets D_k and R_k.

Suppose that $\mathbb{W} \triangleq \{w \in \mathbb{R}^n | f_j^T w \leq g_j, j \in \mathbb{N}_l\}$, where $l \in \mathbb{N}_+$. The fact that $0 \in \text{interior}(\mathbb{W})$ implies that $(f_j, g_j) \in \mathbb{R}^n \times (0,\infty), \forall j \in \mathbb{N}_l$. By definition 9 and by basic properties of the support function, it can be shown that (48) is satisfied if and only if

$$f_j^T \mathcal{A}_{\mathbf{i}_s} w \leq \alpha g_j,\ \forall w \in \mathbb{W} \Leftrightarrow h_{\mathbb{W}}(\mathcal{A}_{\mathbf{i}_s}^T f_j) \leq \alpha g_j \tag{60}$$

for all $\mathbf{i}_s \in \mathcal{I}_s$ and $j \in \mathbb{N}_l$. Furthermore,

$$\begin{aligned} &h_{\mathbb{W}}(\mathcal{A}_{\mathbf{i}_s}^T f_j) \leq \alpha g_j,\ \forall \mathbf{i}_s \in \mathcal{I}_s,\ \forall j \in \mathbb{N}_l \Leftrightarrow \max_{w\in\mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w \leq \alpha g_j,\ \forall \mathbf{i}_s \in \mathcal{I}_s,\ \forall j \in \mathbb{N}_l \\ &\Leftrightarrow \max_{\mathbf{i}_s\in\mathcal{I}_s} \max_{w\in\mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w \leq \alpha g_j,\ \forall j \in \mathbb{N}_l \Leftrightarrow \max_{j\in\mathbb{N}_l} \frac{\max_{\mathbf{i}_s\in\mathcal{I}_s} \max_{w\in\mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w}{g_j} \leq \alpha \end{aligned} \tag{61}$$

Then, equation (61) yields the simple observation that, given $s \in \mathbb{N}^+$,

$$\alpha^o(s) = \max_{j\in\mathbb{N}_l} \frac{\max_{\mathbf{i}_s\in\mathcal{I}_s} \max_{w\in\mathbb{W}} f_j^T \mathcal{A}_{\mathbf{i}_s} w}{g_j} \tag{62}$$

Equation (62) allows us to calculate $\alpha^o(s)$ for a given s without having to explicitly compute the set R_s. Of course, (48) is satisfied if and only if $\alpha^o(s) \in [0,1)$.

The second issue is the calculation of $M(s)$ without having to calculate D_s. Since $\mathbb{W}$ (and D_s) are polytopes, it is appropriate to use the infinity norm for the calculation of $M(s)$. Then:

$$M(s) = \max_{z\in D_s} \|z\|_\infty = \min_\gamma\{\gamma \mid D_s \subseteq \mathbb{B}_\infty^n(\gamma)\}. \tag{63}$$

which is the minimal value of γ for which $D_s \subseteq \mathbb{B}_\infty^n(\gamma)$ holds. The corresponding value of γ, and hence of $M(s)$, can be computed without having to explicitly compute D_s, as shown next.

Combining (18) and (63) it follows that:

$$M(s) = \min_\gamma\{\gamma \mid \text{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s} C_{\mathbf{i}_s}\right) \subseteq \mathbb{B}_\infty^n(\gamma)\}. \tag{64}$$

The set inclusion $\text{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s} C_{\mathbf{i}_s}\right) \subseteq \mathbb{B}_\infty^n(\gamma)$ is equivalent to:

$$C_{\mathbf{i}_s} \subseteq \mathbb{B}_\infty^n(\gamma),\ \forall \mathbf{i}_s \in \mathcal{I}_s \tag{65}$$

Following ideas from [25, Section 4], the simpler set inclusions $C_{\mathbf{i}_s} \subseteq \mathbb{B}_\infty^n(\gamma)$, $\mathbf{i}_s \in \mathcal{I}_s$ are all satisfied if and only if the following inequalities hold:

$$\max_{\mathbf{i}_s\in\mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w \leq \gamma \text{ and } \max_{\mathbf{i}_s\in\mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} (-e_j)^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w \leq \gamma,\ \forall j \in \mathbb{N}_n^+ \tag{66}$$

where e_j is the j^{th} standard basis vector in $\mathbb{R}^n$.

The smallest value for γ can be computed by calculating the maximum of the terms $\max_{\mathbf{i}_s \in \mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w$ and $\max_{\mathbf{i}_s \in \mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} (-e_j)^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w$ for all $j \in \mathbb{N}_n^+$:

$$M(s) = \max_{j\in\mathbb{N}_n^+} \{ \max_{\mathbf{i}_s \in \mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w,\ \max_{\mathbf{i}_s \in \mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} (-e_j)^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w \} \tag{67}$$

The values for $\alpha^o(s)$ and $M(s)$ can be computed from (62) and (67). The results of the above analysis can now be used to formulate Algorithm 1 for the calculation of $D(s, \alpha)$.

Algorithm 1 Computation of an RPI outer ε-approximation of the mCRPI set D_∞

Require: $\mathbb{A}, \mathbb{W}$ and $\varepsilon > 0$

Choose any $s \in \mathbb{N}$ (ideally, set $s \leftarrow 0$).
repeat
  Increment s by one.
  Compute $\alpha^o(s)$ using (62) and set $\alpha \leftarrow \alpha^o(s)$.
  Compute $M(s)$ using (67).
until $\alpha \le \varepsilon/(\varepsilon + M(s))$
Compute D_s using (47)(or (18)) and scale it to give $D(s, \alpha) \triangleq (1-\alpha)^{-1} D_s$.

In order to reduce the computational effort for the calculation of $M(s)$, we observe that it is not necessary to calculate directly $\max_{\mathbf{i}_s \in \mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w$ and $\max_{\mathbf{i}_s \in \mathcal{I}_s} \sum_{l=0}^{s} \max_{w\in\mathbb{W}} (-e_j)^T \mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)} w$ at each iteration of Algorithm 1. Parts of these sums would have been calculated at a previous iteration; thus it is necessary to compute only $\max_{\mathbf{i}_s \in \mathcal{I}_s} \max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)} w$ and $\max_{\mathbf{i}_s \in \mathcal{I}_s} \max_{w\in\mathbb{W}} (-e_j)^T \mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)} w$ and combine this appropriately with already stored results in order to evaluate $M(s)$ in (67) – this requires simple algebraic manipulations and it reduces the computational effort.

Algorithm 1 initially sets s to a fixed value (usually 0) and increases it at each step. The values of α and $M(s)$ are calculated in each iteration using (62) and (67). The algorithm stops when the inequality (59) is satisfied, in which case the *a–priori* specified accuracy $\varepsilon > 0$ has been obtained. The ε-approximation $D(s, \alpha)$ of D_∞ can then be computed using (47)(or (18)) and simple scaling.

The complexity of Algorithm 1 may increase as the state dimension and q increases. However, the algorithm involves the solution of a number of linear programming problems ((62) and (67)) that can be solved more efficiently than working with set calculations. It is also very useful to note that if $\mathbb{W} = \{Ew \mid \|w\|_\infty \le 1\}$, where E is non-singular, then one can compute $\alpha^\circ(s)$ and $M(s)$ without having to resort to solving linear programs, since $\max_{w\in\mathbb{W}} e_j^T \mathcal{A}_{\mathbf{i}_k} w = \|E^T \mathcal{A}_{\mathbf{i}_k}^T e_j\|_1$.

6 Illustrative Examples

To illustrate our results and interesting phenomena occurring when dealing with the reachability analysis of the linear difference inclusions, we provide three simple and constructive $2-D$ examples.

6.1 Example 1

Consider an uncertain discrete-time system that takes the form of (4) with:

$$F_1 = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix},\ F_2 = \begin{bmatrix} 0.8 & 1 \\ 0 & 1 \end{bmatrix},\ G_1 = G_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{68}$$

The additive disturbance set is:

$$\mathbb{W} \triangleq \{w \in \mathbb{R}^2 \mid \|w\|_\infty \le 10\}. \tag{69}$$

The nominal part of the uncertain system (68) can be quadratically stabilized by the state feedback controller:

$$K = [-1.2 \quad -1] \tag{70}$$

Assumption 2 is satisfied with

$$P = \begin{bmatrix} 2.9048 & 0 \\ 0 & 1 \end{bmatrix}, \; \psi = 0.33 \tag{71}$$

The closed loop dynamics are:

$$A_1 = \begin{bmatrix} 0 & 0 \\ -1.2 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.4 & 0 \\ -1.2 & 0 \end{bmatrix} \tag{72}$$

or equivalently:

$$A(a) = \begin{bmatrix} a & 0 \\ -1.2 & 0 \end{bmatrix} \text{ and } a \in \mathbf{a} \triangleq \{a \mid -0.4 \leq a \leq 0\} \tag{73}$$

In order to provide a simple example of a star–shaped RPI set $F(s, \alpha)$, we need to characterize the set sequence $\{R_k^e\}$. We have that $R_0^e = \mathbb{W}$ by definition. The exact explicit characterization of the set R_1^e is obtained as follows:

$$x \in R_1^e \Leftrightarrow x = A(a)w, \; (a, w) \in \mathbf{a} \times \mathbb{W} \tag{74}$$

It follows that:

$$x_1 = aw_1 \text{ and } x_2 = -1.2w_1 \tag{75}$$

where x_i is the i^{th} coordinate of a vector x. From last equation we obtain:

$$w_1 = -x_2/1.2 \text{ and } a = -1.2x_1/x_2 \tag{76}$$

From the bounds on w_1 and a we obtain:

$$-10 \leq -x_2/1.2 \leq 10 \text{ and } -0.4 \leq -1.2x_1/x_2 \leq 0 \tag{77}$$

Solving this set of inequalities we find that:

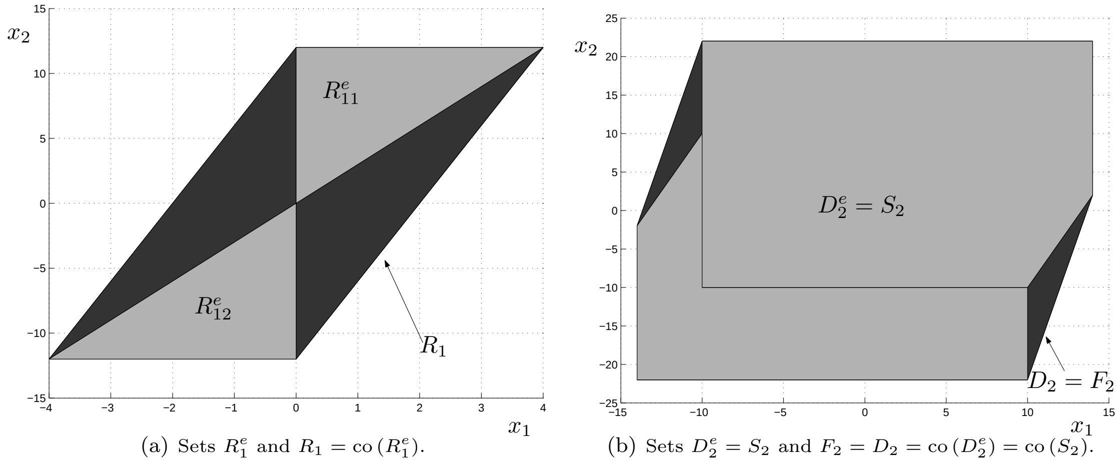
$$\begin{aligned} R_1^e &= R_{11}^e \bigcup R_{12}^e \\ R_{11}^e &\triangleq \{x \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 12, \; 3x_1 - x_2 \leq 0, \; -x_1 \leq 0\} \\ R_{12}^e &\triangleq \{x \in \mathbb{R}^2 \mid -12 \leq x_2 \leq 0, \; -3x_1 + x_2 \leq 0, \; x_1 \leq 0\} \end{aligned} \tag{78}$$

and by (24) $S_2 = R_1^e \oplus R_0^e$. For this particular example it happens to be that $D_2^e = S_2$ where D_2^e is given by (9) ((10)). The sets R_1^e and $R_1 = \text{co}(R_1^e)$ are shown in Figure 1(a), while the sets $D_2^e = S_2$ are shown together with sets $F_2 = D_2 = \text{co}(D_2^e) = \text{co}(S_2)$ in Figure 1(b). Following the same procedure, with necessary changes, it is easy to verify that:

$$\begin{aligned} R_2^e &= R_{21}^e \bigcup R_{22}^e \\ R_{21}^e &\triangleq \{x \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 4.8, \; 3x_1 - x_2 \leq 0, \; -x_1 \leq 0\} \\ R_{22}^e &\triangleq \{x \in \mathbb{R}^2 \mid -4.8 \leq x_2 \leq 0, \; -3x_1 + x_2 \leq 0, \; x_1 \leq 0\} \end{aligned} \tag{79}$$

and consequently the set S_3 is then given by $S_3 = R_2^e \oplus R_1^e \oplus R_0^e$. Unfortunately, the set equality $D_3^e = S_3$ is not true anymore; only set inclusion $D_3^e \subset S_3$ holds. To illustrate this fact we show the sets S_3 and $F_3 = \text{co}(S_3)$ in Figure 2 and a vector $p = [-11.6 \; 17.2]^T$ that satisfies $p \in S_3$. To verify that $p \in S_3$ we note that we have (for example):

$$p = \begin{bmatrix} -1.6 \\ -4.8 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \end{bmatrix} + \begin{bmatrix} -10 \\ 10 \end{bmatrix} \text{ and } \begin{bmatrix} -1.6 \\ -4.8 \end{bmatrix} \in R_2^e, \; \begin{bmatrix} 0 \\ 12 \end{bmatrix} \in R_1^e, \; \begin{bmatrix} -10 \\ 10 \end{bmatrix} \in R_0^e. \tag{80}$$



(a) Sets R_1^e and $R_1 = \text{co}\,(R_1^e)$.

(b) Sets $D_2^e = S_2$ and $F_2 = D_2 = \text{co}\,(D_2^e) = \text{co}\,(S_2)$.

Figure 1: Sets R_1^e, R_1, $D_2 = F_2$ and $D_2^e = S_2$

To show that D_3^e is not equal to S_3 (but only $D_3^e \subset S_3$) we show that there exists no uncertainty realization resulting in vector p by contradiction. Suppose that $p \in D_3^e$ in which case we would have:

$$p = A(a(2))A(a(1))w(0) + A(a(2))w(1) + w(2),\ \ a(1), a(2) \in \mathbf{a},\ \ w(0), w(1), w(2) \in \mathbb{W} \tag{81}$$

The last equation is equivalent to:

$$-11.6 = a(2)a(1)w_1(0) + a(2)w_1(1) + w_1(2) \tag{82a}$$
$$17.2 = -1.2a(1)w_1(0) - 1.2w_1(1) + w_2(2) \tag{82b}$$

From (82b) it follows that:

$$a(1)w_1(0) + w_1(1) = \frac{w_2(2) - 17.2}{1.2} \tag{83}$$

which when substituted in (82a) yields:

$$-11.6 - w_1(2) = \frac{a(2)(w_2(2) - 17.2)}{1.2} \tag{84}$$

Solving (84) for $a(2)$ we have:

$$a(2) = \frac{13.92 + 1.2w_1(2)}{17.2 - w_2(2)} \tag{85}$$

Using bounds on $a(2)$ we further have:

$$-0.4 \leq \frac{13.92 + 1.2w_1(2)}{17.2 - w_2(2)} \leq 0 \tag{86}$$

or equivalently, after some elementary algebraic manipulations:

$$0 \leq \frac{104 + 4w_1(2)}{34.4 - 2w_2(2)} \text{ and } \frac{69.6 + 6w_1(2)}{34.4 - 2w_2(2)} \leq 0 \tag{87}$$

Since $-10 \leq w_2(2) \leq 10$ it follows that $14.4 \leq 34.4 - 2w_2(2) \leq 54.4$ so that the last inequalities reduce to:

$$0 \leq 104 + 4w_1(2) \text{ and } 69.6 + 6w_1(2) \leq 0 \text{ or equivalently } -26 \leq w_1(2) \leq -11.6 \tag{88}$$

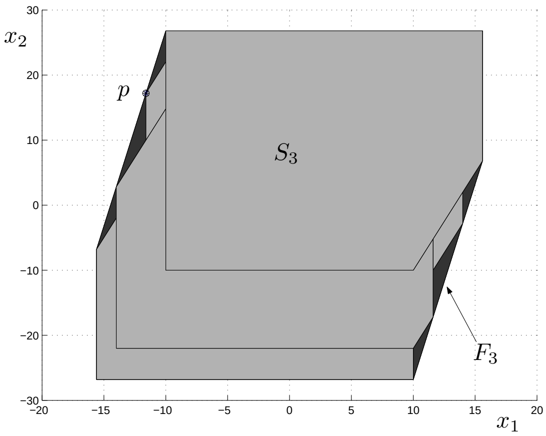


Figure 2: Set S_3 and vector p

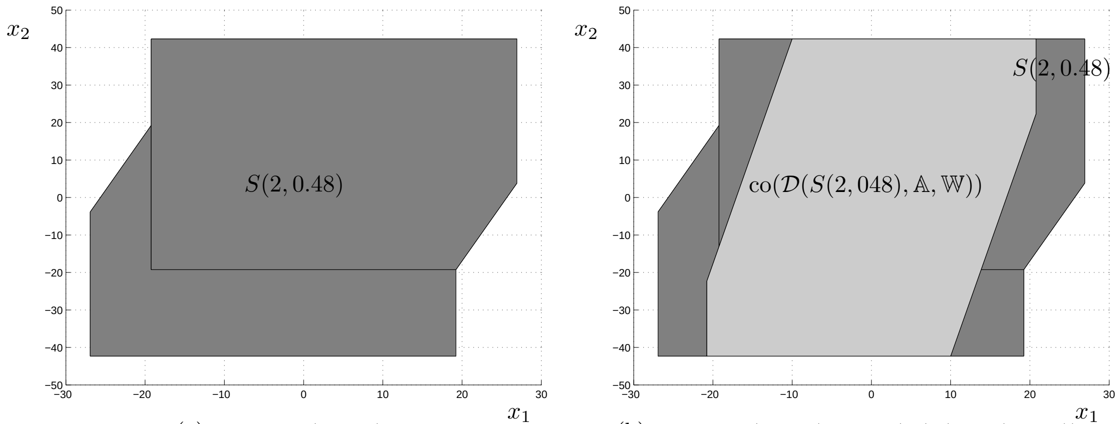
which contradicts the bounds on $w_1(2)$ $(-10 \leq w_1(2) \leq 10)$; consequently $p \notin D_3^e$ and D_3^e is not equal to S_3. To illustrate a star–shaped RPI set $S(s, \alpha)$, we observe that for instance:

$$R_2^e \subseteq 0.48\mathbb{W} \tag{89}$$

Thus by choosing $(s, \alpha) = (2, 0.48)$ (in fact any α such that $0.48 \leq \alpha < 1$ can be used) we construct the following RPI set:

$$S(2, 0.48) = 0.52^{-1} S_2 \tag{90}$$

The set $S(2, 0.48)$ together with $\text{co}(\mathcal{D}(S(2, 048), \mathbb{A}, \mathbb{W}))$ is shown in Figure 3. It is consequently true that $F(2, 0.48)$



(a) RPI Set $S(2, 0.48)$.

(b) RPI Set $S(2, 0.48)$ and $\text{co}(\mathcal{D}(S(2, 048), \mathbb{A}, \mathbb{W}))$.

Figure 3: Sets $S(2, 0.48)$ and $\text{co}(\mathcal{D}(S(2, 048), \mathbb{A}, \mathbb{W}))$

is also an RPI set since $F(2, 0.48) = \text{co}(S(2, 0.48))$.

We finally remark that for this particular example:

$$
\begin{aligned}
R_k^e &= R_{k1}^e \bigcup R_{k2}^e \\
R_{k1}^e &\triangleq \{x \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 12(0.4)^{k-1},\ 3x_1 - x_2 \leq 0,\ -x_1 \leq 0\} \\
R_{k2}^e &\triangleq \{x \in \mathbb{R}^2 \mid -12(0.4)^{k-1} \leq x_2 \leq 0,\ -3x_1 + x_2 \leq 0,\ x_1 \leq 0\}
\end{aligned} \tag{91}
$$

Consequently, $R_k^e \to \{0\}$ exponentially fast (in the Hausdorff Metric) as $k \to \infty$ as expected by Theorem 1.

6.2 Example 2

Our second example is based on a very simple construction and illustrates that there are degenerate cases for which $D_\infty^e = D_\infty = S_\infty = F_\infty$ and all of these sets are convex. We observe that for this particular example Assumptions 1 and 2 are violated but despite this fact the sets $D_\infty^e = D_\infty = S_\infty = F_\infty$ are well defined. Consider the system, similar to the one in sub–section 6.1:

$$
A(a,b) = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \tag{92}
$$

where (a,b) is an uncertain pair of bounded scalars (the uncertainty can be arbitrarily large), i.e. $\underline{a} \leq a \leq \bar{a}$ and $\underline{b} \leq b \leq \bar{b}$ where $\underline{a} \leq \bar{a}$ and $\underline{b} \leq \bar{b}$ with scalars $\underline{a}$, $\bar{a}$, $\underline{b}$ and $\bar{b}$ being arbitrarily large but bounded. The disturbance set is:

$$
\mathbb{W} = \{w \in \mathbb{R}^2 \mid w_1 = 0,\ c \leq w_2 \leq d\},\ c \leq 0 \leq d \tag{93}
$$

where (c,d) is an arbitrary pair of bounded scalars (arbitrarily large). In this case, the vectors belonging to $\mathbb{W}$ lie in the null–space of $A(a,b)$ for arbitrary (a,b); thus it is trivial to observe that:

$$
D_\infty^e = D_\infty = S_\infty = F_\infty = \mathbb{W} \tag{94}
$$

This example illustrates that it is, at least in principle, possible to encounter examples for which surprising set equality $D_\infty^e = D_\infty = S_\infty = F_\infty$ holds and that all of these sets are convex. However, such a construction is possible only for the examples of particular structure; thus this is not a generic case.

6.3 Example 3

Our third example illustrates that $D_k \subset F_k$. Consider the system (4) with

$$
F_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1.1 \end{bmatrix},\ F_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0.9 \end{bmatrix},\ G_1 = G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}'
$$

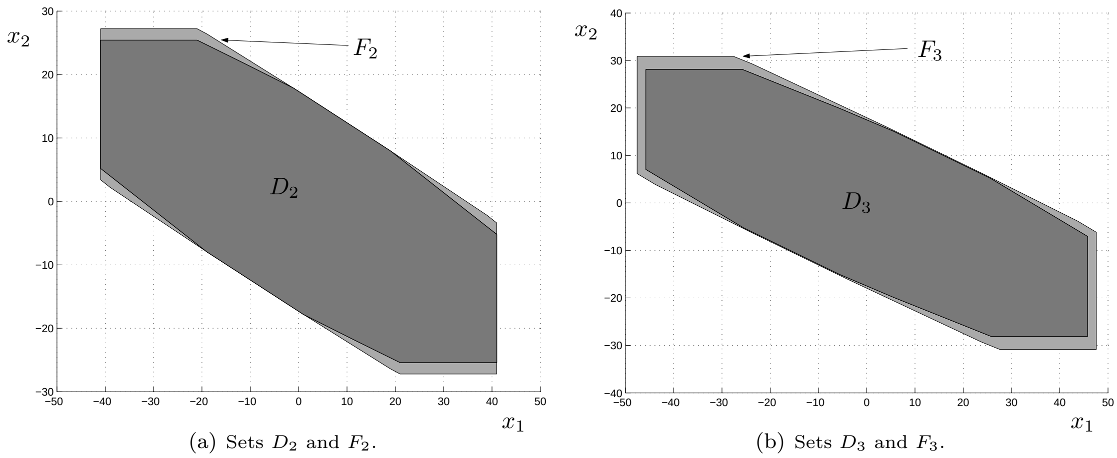
and

$$
\mathbb{W} \triangleq \{w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 10\}.
$$

The feedback controller $K = \begin{bmatrix} -1.4936 & -1.5073 \end{bmatrix}$ quadratically stabilizes the uncertain system with

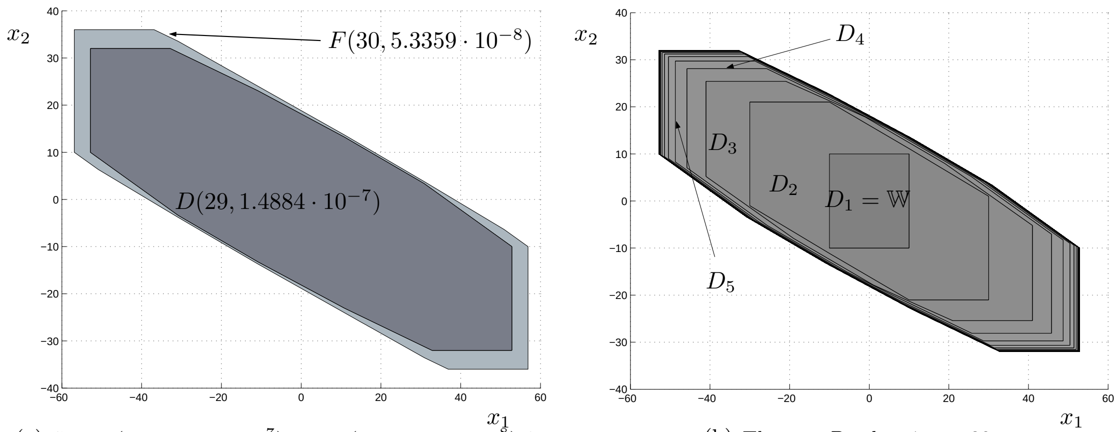
$$
P = \begin{bmatrix} 0.07133 & 0.046751 \\ 0.046751 & 0.094124 \end{bmatrix},\ \psi = 0.22
$$

Figure 4 shows the sets D_2, D_3 and F_2, F_3 and illustrates that $D_2 \subset F_2$ and $D_3 \subset F_3$ indicating that $D_k \subset F_k$ is a generic case. Figure 5(a) illustrates that approximations $D(29, 1.4884 \cdot 10^{-7})$ and $F(30, 5.3359 \cdot 10^{-8})$ of D_∞ and F_∞, respectively, satisfy $D(29, 1.4884 \cdot 10^{-7}) \subset F(30, 5.3359 \cdot 10^{-8})$ indicating that $D_\infty \subset F_\infty$. Both approximations where obtained by setting value $\epsilon = 10^{-5}$. Figure 5(b) shows the set sequence $\{D_k\}$ (for $k = 1, \ldots, 29$) and indicates that the set sequence $\{D_k\}$ is convergent.



(a) Sets D_2 and F_2. (b) Sets D_3 and F_3.

Figure 4: Sets D_2, D_3 and F_2, F_3.



(a) Sets $D(29, 1.4884 \cdot 10^{-7})$ and $F(30, 5.3359 \cdot 10^{-8})$ for $\epsilon = 10^{-5}$. (b) The sets D_k, $k = 1, \ldots, 29$.

Figure 5: Outer RPI ϵ-approximations and D_k, $k = 1, \ldots, 29$.

7 Conclusions

The novel results reported in this paper further extend the existing research for the computation and approximation of the mCRPI set for autonomous linear discrete-time systems [25]. The results have been extended to address the more general and difficult case of *linear difference inclusions.* Three families of RPI sets for linear difference inclusion have been characterized. The existence and characterization of basic star–shaped RPI sets for linear difference inclusion is established for the first time. A relevant contribution is a method for the computation of the outer RPI ε-approximation, of the mCRPI set for *linear difference inclusions*, for an *a priori* given $\varepsilon > 0$. The proposed method is efficient in that it involves the computation of a number of linear programming problems and simple algebraic calculations instead of less tractable calculations with sets. It is in principle possible to further improve computational aspects and this extension is a subject of current research. Several numerical examples are also provided to illustrate interesting phenomena occurring when dealing with robust positive invariance issues for linear difference inclusions.

The results presented in this paper can be exploited in robust control of linear difference inclusions subject to constraints and additive but bounded disturbances [5, 7, 11].

APPENDIX A – Necessary technical and preliminary results

APPENDIX A1 – Some Properties of Linear Difference Inclusion (1) and its One–Step Forward Reachable Set (7)

We establish only some elementary properties of $\mathcal{D}(X, \mathbb{A}, \mathbb{W})$, that easily follow from Definition of $\mathcal{D}(X, \mathbb{A}, \mathbb{W})$ and are necessary to simplify our proofs; we also remark that a number of additional properties is easily established.

Lemma 1 *Let X and Y be two arbitrary non–empty sets in $\mathbb{R}^n$. Then:*

(i) Property X: $\mathcal{D}(X \oplus Y, \mathbb{A}, \mathbb{W}) \subseteq \mathcal{D}(X, \mathbb{A}, \mathbb{W}) \oplus \mathcal{D}(Y, \mathbb{A}, \{0\})$,

(ii) Property Y: $\mathcal{D}(\{0\} \oplus Y, \mathbb{A}, \mathbb{W}) = \mathcal{D}(\{0\}, \mathbb{A}, \mathbb{W}) \oplus \mathcal{D}(Y, \mathbb{A}, \{0\})$,

(iii) Property Z: $\mathcal{D}(X \oplus Y, \mathbb{A}, \{0\}) \subseteq \mathcal{D}(X, \mathbb{A}, \{0\}) \oplus \mathcal{D}(Y, \mathbb{A}, \{0\})$,

Proof:

(i) For arbitrary $z \in \mathcal{D}(X \oplus Y, \mathbb{A}, \mathbb{W})$ holds that $z = A(x + y) + w$ where $(x, y, A, w) \in X \times Y \times \mathrm{co}(\mathbb{A}) \times \mathbb{W}$. We have:

$$z = A(x + y) + w = Ax + Ay + w = Ax + w + Ay$$

since $z_1 = Ax + w \in \mathcal{D}(X, \mathbb{A}, \mathbb{W})$ and $z_2 = Ay \in \mathcal{D}(Y, \mathbb{A}, \{0\})$ it follows that $\mathcal{D}(X \oplus Y, \mathbb{A}, \mathbb{W}) \subseteq \mathcal{D}(X, \mathbb{A}, \mathbb{W}) \oplus \mathcal{D}(Y, \mathbb{A}, \{0\})$.

(ii) This property follows from the Definition of $\mathcal{D}(X, \mathbb{A}, \mathbb{W})$.

(iii) This property follows from (i) with $\mathbb{W} = \{0\}$.

Q.e.D.

*W*e also recall that it is always true that:

(i) $\alpha(X \oplus Y) = \alpha X \oplus \alpha Y$ for any two arbitrary (non–empty) sets X and Y in $\mathbb{R}^n$ and an arbitrary $\alpha \in \mathbb{R}$.

(ii) $\mathrm{co}(X \cup Y) = \mathrm{co}(\mathrm{co}(X) \cup \mathrm{co}(Y))$ for any two arbitrary (non–empty) sets X and Y in $\mathbb{R}^n$.

APPENDIX A2 – Some Additional Technical Results

The following result, that can be stated in stronger form, is clear but it is provided here for a sake of completeness.

Lemma 2 *Let $\{\mathcal{S}_k \subset \mathbb{R}^n\}$ be a Cauchy sequence of compact sets such that $\mathcal{S}_k \subseteq \mathcal{S}_{k+1}$, $\forall k \in \mathbb{N}$. Then:*

(i) There exists a compact set $\bar{\mathcal{S}}$ that is a limit of the set sequence $\{\mathcal{S}_k\}$ in the Hausdorff metric sense,

(ii) If $\mathcal{S}_k^c \triangleq \mathrm{co}\,(\mathcal{S}_k)\,, \forall k \in \mathbb{N}$, then set sequence $\{\mathcal{S}_k^c\}$ is also a Cauchy sequence of compact sets , its limit $\bar{\mathcal{S}}^c$ in the Hausdorff metric exists and it satisfies $\bar{\mathcal{S}}^c = \mathrm{co}\left(\bar{\mathcal{S}}\right)$.

Proof:

(i) This claim follows from the facts that: (a) a family of compact sets, each of which is a subset of $\mathbb{R}^n$, equipped with Hasudorff metric is a complete metric space [35] and (b) the sequence $\{\mathcal{S}_k\}$ is a Cauchy sequence. Hence, since (i) is true, $\{\mathcal{S}_k\}$ has a limit $\bar{\mathcal{S}}$ in the Hausdorff metric sense that is an element of the space.

(ii) This claim is a direct consequence of the definition of the set sequence $\{\mathcal{S}_k^c\}$.

Q.e.D.

The following result is a direct consequence of Theorem 1 and is provided here for sake of completeness.

Theorem 7 *Suppose Assumptions 1 and 2 hold. Then the set sequence $\{S_k\}$ defined by (24) satisfies :*
(i) There exist $\theta \in (0,1)$ and $\mu < \infty$ such that $S_k \subseteq S_{k+1} \subseteq \mathcal{S}_k \oplus \theta^k \mathbb{B}_p^n(\mu)$ for all $k \in \mathbb{N}$,
(ii) there exists a compact set S_∞ such that $\mathcal{H}(S_\infty, S_k) \to 0$ as $k \to \infty$.

Proof: The proof of this result follows the same arguments as the proof of Theorem 1 given in the appendix B2. Q.e.D.

APPENDIX B – Proofs

APPENDIX B1 – Proof of Proposition 1

First we show that

$$D_k^e \subseteq D_{k+1}^e \subseteq D_k^e \oplus R_k^e$$

by principle of mathematical induction. We have $D_1^e = D_0^e \oplus R_0^e$. We assume then, that $D_k^e \subseteq D_{k+1}^e \subseteq D_k^e \oplus R_k^e$ and we show that $D_{k+2}^e \subseteq D_{k+1}^e \oplus R_{k+1}^e$. Observe that $\mathcal{D}(X, \mathrm{co}(\mathbb{A}), \mathbb{W}) \subseteq \mathcal{D}(Y, \mathrm{co}(\mathbb{A}), \mathbb{W})$ for any sets $X \subseteq Y \subset \mathbb{R}^n$ and recall Property X from Appendix A1. Then

$$D_{k+2}^e = \mathcal{D}(D_{k+1}^e, \mathbb{A}, \mathbb{W}) \subseteq \mathcal{D}(D_k^e \oplus R_k^e, \mathbb{A}, \mathbb{W}) \subseteq \mathcal{D}(D_k^e, \mathbb{A}, \mathbb{W}) \oplus \mathcal{D}(R_k^e, \mathbb{A}, \{0\}) = D_{k+1}^e \oplus R_{k+1}^e$$

and the first claim is verified since clearly $D_k^e \subseteq D_{k+1}^e$, $\forall k \in \mathbb{N}$.

To establish the second claim we only show that D_k^e is a basic star–shaped set since the same arguments are easily exploited to verify that R_k^e is a basic star–shaped set. In order to show that D_k^e is a basic star–shaped set we only have to show, by recalling Definition 3 (with $w_c = 0$), that if $z \in D_k^e$ then $\lambda z \in D_k^e$ for any $\lambda \in (0,1]$. Since D_k^e is a set of the forward reachable tube from the origin of the linear diference inclusion 1, then for every $z \in D_k^e$ there exists (at least one feasible combination) $(A(i), w(i)) \in \mathrm{co}(\mathbb{A}) \times \mathbb{W}$, $i \in \mathbb{N}_{k-1}^+$ such that

$$z = A(k-1)A(k-2)\dots A(2)A(1)w(0) + A(k-1)A(k-2)\dots A(2)w(1) + \dots A(k-1)w(k-2) + w(k-1)$$

and we proceed with the proof assuming that an appropriate selection of this realization is made (for instance minimum–two norm uncertainty realization). Then, for any $\lambda \in (0,1]$

$$\lambda z = A(k-1)A(k-2)\dots A(2)A(1)\lambda w(0) + A(k-1)A(k-2)\dots A(2)\lambda w(1) + \dots A(k-1)\lambda w(k-2) + \lambda w(k-1)$$

Since $\lambda \in (0,1]$ and $\mathbb{W}$ is a convex set then $\lambda w(i) \in \mathbb{W}$. If we set $w^o(i) = \lambda w(i)$ we have $w^o(i) \in \mathbb{W}$, $(A(i), w^o(i)) \in \mathrm{co}(\mathbb{A}) \times \mathbb{W}$ and

$$\lambda z = A(k-1)A(k-2)\dots A(2)A(1)w^o(0) + A(k-1)A(k-2)\dots A(2)w^o(1) + \dots A(k-1)w^o(k-2) + w^o(k-1)$$

Hence, $\lambda z \in D_k^e$ and the second claim is verified since $\lambda \in (0,1]$ is arbitrary which completes our proof.

APPENDIX B2 – Proof of Theorem 1

This claim is verified as follows:

(i) This fact follows from (13) and Assumptions 1 and 2,

(ii) The part (i) implies that $\mathcal{H}(D^e_{k+1}, D^e_k) \leq \mu\theta^k$ which in turns imply that

$$\lim_{k\to\infty} \max_{m\geq 0} \mathcal{H}(D^e_{k+m}, D^e_k) \leq \theta^k \mu (1-\theta)^{-1}$$

Hence, since $\mu < \infty$ and $\theta \in (0,1)$, it follows that there exists a compact set D^e_∞ such that $\mathcal{H}(D^e_\infty, D^e_k) \to 0$ as $k \to \infty$.

APPENDIX B3 – Proof of Proposition 2

To establish this claim we resort to the principle of mathematical induction. Suppose that for some $k \in \mathbb{N}$ we have:

$$D_k = \mathrm{co}\left(\bigcup_{\mathbf{i}_{k-1}\in\mathcal{I}_{k-1}} C_{\mathbf{i}_{k-1}}\right)$$

where the sets $C_{\mathbf{i}_{k-1}},\ \mathbf{i}_{k-1} \in \mathcal{I}_{k-1}$ are given by:

$$C_{\mathbf{i}_{k-1}} \triangleq \bigoplus_{l=0}^{k-1} \mathcal{A}_{\mathbf{j}_{k-1-l}(\mathbf{i}_{k-1})} \mathbb{W}$$

Direct calculation yields:

$$\begin{aligned}
D_{k+1} &= \mathrm{co}\left(\bigcup_{i\in\mathbb{N}_q^+} A_i D_k\right) \oplus \mathbb{W} = \mathrm{co}\left(\bigcup_{i\in\mathbb{N}_q^+} A_i\, \mathrm{co}\left(\bigcup_{\mathbf{i}_{k-1}\in\mathcal{I}_{k-1}} C_{\mathbf{i}_{k-1}}\right)\right) \oplus \mathbb{W} \\
&= \mathrm{co}\left(\bigcup_{i\in\mathbb{N}_q^+} \mathrm{co}\left(A_i \bigcup_{\mathbf{i}_{k-1}\in\mathcal{I}_{k-1}} C_{\mathbf{i}_{k-1}}\right)\right) \oplus \mathbb{W} = \mathrm{co}\left(\bigcup_{i\in\mathbb{N}_q^+} \mathrm{co}\left(\bigcup_{\mathbf{i}_{k-1}\in\mathcal{I}_{k-1}} A_i C_{\mathbf{i}_{k-1}}\right)\right) \oplus \mathbb{W} \\
&= \mathrm{co}\left(\mathrm{co}\left(\bigcup_{i\in\mathbb{N}_q^+} \bigcup_{\mathbf{i}_{k-1}\in\mathcal{I}_{k-1}} A_i C_{\mathbf{i}_{k-1}}\right)\right) \oplus \mathbb{W} = \mathrm{co}\left(\mathrm{co}\left(\bigcup_{(i,\mathbf{i}_{k-1})\in\mathbb{N}_q^+\times\mathcal{I}_{k-1}} A_i C_{\mathbf{i}_{k-1}}\right)\right) \oplus \mathbb{W} \\
&= \mathrm{co}\left(\mathrm{co}\left(\bigcup_{(i,\mathbf{i}_{k-1})\in\mathbb{N}_q^+\times\mathcal{I}_{k-1}} A_i C_{\mathbf{i}_{k-1}}\right) \oplus \mathbb{W}\right) = \mathrm{co}\left(\bigcup_{(i,\mathbf{i}_{k-1})\in\mathbb{N}_q^+\times\mathcal{I}_{k-1}} A_i C_{\mathbf{i}_{k-1}} \oplus \mathbb{W}\right) \\
&= \mathrm{co}\left(\bigcup_{(i,\mathbf{i}_{k-1})\in\mathbb{N}_q^+\times\mathcal{I}_{k-1}} \left(A_i C_{\mathbf{i}_{k-1}} \oplus \mathbb{W}\right)\right) = \mathrm{co}\left(\bigcup_{\mathbf{i}_k\in\mathcal{I}_k} C_{\mathbf{i}_k}\right)
\end{aligned}$$

The proof is completed by realizing that induction base is trivially true.

APPENDIX B4 – Proof of Proposition 3

We prove that $D_k = \mathrm{co}(D^e_k)$ by induction. Hence, we first show that if $D_k = \mathrm{co}(D^e_k)$ then $D^e_{k+1} \subseteq D_{k+1}$, because this means $\mathrm{co}(D^e_{k+1}) \subseteq D_{k+1}$ since D_{k+1} is convex. We subsequently show that $D_{k+1} \subseteq \mathrm{co}(D^e_{k+1})$ (when $D_k = \mathrm{co}(D^e_k)$) completing the proof.

Suppose that for some $k \in \mathbb{N}$ we have $\text{co}(D_k^e) = D_k$. Then for any $y \in D_{k+1}^e$ there exist $(x, A, w) \in D_k^e \times \text{co}(\mathbb{A}) \times \mathbb{W}$ such that

$$y = Ax + w = (\sum_{i=1}^{q} \lambda_i A_i)x + w = \sum_{i=1}^{q} \lambda_i (A_i x) + w$$

where $\lambda = (\lambda_1, \ldots, \lambda_q) \in \Lambda$ defined in (3) (these are standard convex multipliers). Since $x \in D_k^e \subseteq \text{co}(D_k^e) = D_k$ then $A_i x \in \text{co}\left(\bigcup_{i \in \mathbb{N}_q^+} A_i D_k\right)$ and, furthermore, $\sum_{i=1}^{q} \lambda_i A_i x \in \text{co}\left(\bigcup_{i \in \mathbb{N}_q^+} A_i D_k\right)$. Finally

$$y = \sum_{i=1}^{q} \lambda_i (A_i x) + w \in \text{co}\left(\bigcup_{i \in \mathbb{N}_q^+} A_i D_k\right) + \mathbb{W} = D_{k+1}$$

which implies that $D_{k+1}^e \subseteq D_{k+1}$ and the first part of the proof is completed.

We now proceed to prove $D_{k+1} \subseteq \text{co}(D_{k+1}^e)$. Recall from Proposition 2 the alternative definition of D_k in equation (18) and the definition of $C_{\mathbf{i}_k}$ (19) and (20). It is obvious from the definition of $C_{\mathbf{i}_{k+1}}$ that for any $x \in C_{\mathbf{i}_{k+1}}$ there exists $w(l) \in \mathbb{W}$, $l \in \mathbb{N}_{k+1}$ such that

$$x = \sum_{l=0}^{k+1} \mathcal{A}_{\mathbf{j}_{k+1-l}(\mathbf{i}_{k+1})} w(l)$$

which shows that every $x \in C_{\mathbf{i}_{k+1}}$ is a reachable state from the origin in $k+1$ steps. By definition of D_{k+1}^e, it follows that $x \in D_{k+1}^e$ and hence $C_{\mathbf{i}_{k+1}} \subseteq D_{k+1}^e$, consequently $\cup_{\mathbf{i}_{k+1} \in \mathcal{I}_{k+1}} C_{\mathbf{i}_{k+1}} \subseteq D_{k+1}^e$ and $\text{co}(\bigcup_{\mathbf{i}_{k+1} \in \mathcal{I}_{k+1}} C_{\mathbf{i}_{k+1}}) \subseteq \text{co}(D_{k+1}^e)$ which yields:

$$D_{k+1} = \text{co}(\bigcup_{\mathbf{i}_{k+1} \in \mathcal{I}_{k+1}} C_{\mathbf{i}_{k+1}}) \subseteq \text{co}(D_{k+1}^e)$$

and the second part of the proof is concluded which completes our proof since clearly induction base is true $D_0 = D_0^e$ implies $D_1 = D_1^e$ by definition of the set sequences $\{D_k\}$ and $\{D_k^e\}$.

APPENDIX B5 – Proof of Proposition 4

We prove the claim that $D_k^e \subseteq S_k$ for all $k \in \mathbb{N}$ by mathematical induction. Obviously, by definition $D_0^e \subseteq S_0$ and $D_1^e \subseteq S_1$. Then, assume that for some $k \in \mathbb{N}$ we have:

$$D_k^e \subseteq S_k$$

Then by Proposition 1,

$$D_{k+1}^e = \mathcal{D}(D_k^e, \mathbb{A}, \mathbb{W}) \subseteq D_k^e \oplus R_k^e$$

and consequently, since $D_k^e \subseteq S_k$,

$$D_{k+1}^e \subseteq S_k \oplus R_k^e = S_{k+1}$$

and the proof is completed.

APPENDIX B6 – Proof of Theorem 2

The fact that the set $\mathcal{P}_S$ defined in (31) is non–empty follows directly from Assumptions 1 and 2. The claim that the set $S(s, \alpha)$ is a basic star–shaped set is a consequence of the fact that S_s is a basic star–shaped set and so is $(1-\alpha)^{-1} S_s$. The fact that $D_\infty^e \subseteq S_\infty \subseteq S(s, \alpha)$ follows from Proposition 4. In order to verify our result it remains to show that $S(s, \alpha)$ is an RPI set for linear difference inclusion for $(s, \alpha) \in \mathcal{P}_S$. We have:

$$\mathcal{D}(S(s, \alpha), \mathbb{A}, \mathbb{W}) = \mathcal{D}((1-\alpha)^{-1} S_s, \mathbb{A}, \mathbb{W}) = \mathcal{D}((1-\alpha)^{-1} \bigoplus_{j=0}^{s-1} R_j^e, \mathbb{A}, \mathbb{W})$$

By Property X established in Appendix A1 we have:

$$\mathcal{D}((1-\alpha)^{-1}\bigoplus_{j=0}^{s-1} R_j^e, \mathbb{A}, \mathbb{W}) \subseteq \mathcal{D}((1-\alpha)^{-1}\{0\}, \mathbb{A}, \mathbb{W}) \oplus \bigoplus_{j=0}^{s-1} \mathcal{D}((1-\alpha)^{-1}R_j^e, \mathbb{A}, \{0\})$$
$$\subseteq \mathbb{W} \oplus \bigoplus_{j=1}^{s}(1-\alpha)^{-1}R_j^e = \mathbb{W} \oplus (1-\alpha)^{-1}R_s^e \oplus \bigoplus_{j=1}^{s-1}(1-\alpha)^{-1}R_j^e$$

However, since $(s,\alpha) \in \mathcal{P}_S$ we have that $R_s^e \subseteq \alpha\mathbb{W}$ and consequently:

$$\mathbb{W} \oplus (1-\alpha)^{-1}R_s^e \subseteq \mathbb{W} \oplus (1-\alpha)^{-1}\alpha\mathbb{W} = (1-\alpha)^{-1}\mathbb{W}$$

Hence,

$$\mathcal{D}(S(s,\alpha), \mathbb{A}, \mathbb{W}) \subseteq \mathbb{W} \oplus (1-\alpha)^{-1}R_s^e \oplus \bigoplus_{j=1}^{s-1}(1-\alpha)^{-1}R_j^e \subseteq (1-\alpha)^{-1}\mathbb{W} \oplus \bigoplus_{j=1}^{s-1}(1-\alpha)^{-1}R_j^e$$

and

$$\mathcal{D}(S(s,\alpha), \mathbb{A}, \mathbb{W}) \subseteq \bigoplus_{j=0}^{s-1}(1-\alpha)^{-1}R_j^e = S(s,\alpha)$$

Thus $S(s,\alpha)$ is an RPI set.

APPENDIX B7 – Proof of Proposition 5

Arguments similar to those employed in the proof of Proposition 3 establish this claim.

APPENDIX B8 – Proof of Proposition 6

We claim that $F_k = \mathrm{co}(S_k)$. This is true because, for any finite $k \in \mathbb{N}$, we have by Proposition 5 and (16):

$$F_k = \bigoplus_{j=0}^{k-1} R_j = \bigoplus_{j=0}^{k-1} \mathrm{co}(R_j^e) = \mathrm{co}\left(\bigoplus_{j=0}^{k-1} R_j^e\right) = \mathrm{co}(S_k)$$

Note that this claim remains valid for the limit $F_\infty = \mathrm{co}(S_\infty)$, if Assumptions 1 and 2 hold, due to Theorems 7 and Lemma 2. This completes our proof.

APPENDIX B9 – Proof of Theorem 3

Observing that conditions (30) and (42) are equivalent (when Assumptions 1 and 2 hold) and that $F_s = \mathrm{co}(S_s)$, we conclude that $F(s,\alpha) = \mathrm{co}(S(s,\alpha))$. Moreover, since $S(s,\alpha)$ is an RPI set then, by Theorem 2, $F(s,\alpha) = \mathrm{co}(S(s,\alpha))$ is also an RPI set such that $D_\infty^e \subseteq F_\infty \subseteq F(s,\alpha)$.

APPENDIX B10 – Proof of Theorem 4

The claims that the set $\mathcal{P}_D$ (defined in (51)) is non–empty and that $D(s,\alpha)$, $(s,\alpha) \in \mathcal{P}_D$ is a C set are direct consequences of Assumptions 1 and 2 and Definition of the set $D(s,\alpha)$. We prove that, given $(s,\alpha) \in \mathcal{P}_D$, the set $D(s,\alpha)$ is an RPI set for the linear difference inclusion (1); the claim that $D_\infty^e \subseteq D_\infty \subseteq D(s,\alpha)$ is clearly true by definition of the sets D_∞^e, D_∞ and $D(s,\alpha)$. Consider a set $D(s,\alpha)$, $(s,\alpha) \in \mathcal{P}_D$. We proceed as follows:

$$\mathcal{D}(D(s,\alpha), \mathbb{A}, \mathbb{W}) \subseteq \mathrm{co}\left(\bigcup_{i\in\mathbb{N}_q^+} A_i D(s,\alpha)\right) \oplus \mathbb{W}$$

But $D(s,\alpha)$ is defined by:

$$D(s,\alpha) = (1-\alpha)^{-1}\operatorname{co}\left(\bigcup_{\mathbf{i}_{s-1}\in\mathcal{I}_{s-1}} C_{\mathbf{i}_{s-1}}\right)$$

where the sets $C_{\mathbf{i}_{s-1}},\ \mathbf{i}_{s-1}\in\mathcal{I}_{s-1}$ are given by:

$$C_{\mathbf{i}_{s-1}} \triangleq \bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-1-l}(\mathbf{i}_{s-1})}\mathbb{W}$$

Following the arguments employed in the proof of Proposition 2, we have

$$\begin{aligned}
\mathcal{D}(D(s,\alpha),\mathbb{A},\mathbb{W}) &\subseteq \operatorname{co}\left(\bigcup_{i\in\mathbb{N}_q^+} A_i(1-\alpha)^{-1}\operatorname{co}\left(\bigcup_{\mathbf{i}_{s-1}\in\mathcal{I}_{s-1}}\bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-1-l}(\mathbf{i}_{s-1})}\mathbb{W}\right)\right)\oplus\mathbb{W}\\
&= \operatorname{co}\left(\bigcup_{(i,\mathbf{i}_{s-1})\in\mathbb{N}_q^+\times\mathcal{I}_{s-1}}(1-\alpha)^{-1}A_i\bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-1-l}(\mathbf{i}_{s-1})}\mathbb{W}\right)\oplus\mathbb{W}\\
&= \operatorname{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s}(1-\alpha)^{-1}\bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)}\mathbb{W}\right)\oplus\mathbb{W}\\
&= \operatorname{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s}(1-\alpha)^{-1}\bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)}\mathbb{W}\oplus\mathbb{W}\right)\\
&= \operatorname{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s}\left((1-\alpha)^{-1}\bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)}\mathbb{W}\oplus\mathbb{W}\right)\right)\\
&= \operatorname{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s}\left((1-\alpha)^{-1}\bigoplus_{l=1}^{s-1}\mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)}\mathbb{W}\oplus(1-\alpha)^{-1}\mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)}\mathbb{W}\oplus\mathbb{W}\right)\right)
\end{aligned}$$

Since $(s,\alpha)\in\mathcal{P}_D$ we have:

$$(1-\alpha)^{-1}\mathcal{A}_{\mathbf{j}_s(\mathbf{i}_s)}\mathbb{W}\oplus\mathbb{W}\subseteq(1-\alpha)^{-1}\alpha\mathbb{W}\oplus\mathbb{W}=(1-\alpha)^{-1}\mathbb{W}$$

for all $\mathbf{i}_s\in\mathcal{I}_s$. Combining the last two equations we obtain:

$$\begin{aligned}
\mathcal{D}(D(s,\alpha),\mathbb{A},\mathbb{W}) &\subseteq \operatorname{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s}\left((1-\alpha)^{-1}\bigoplus_{l=1}^{s-1}\mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)}\mathbb{W}\oplus(1-\alpha)^{-1}\mathbb{W}\right)\right)\\
&= \operatorname{co}\left(\bigcup_{\mathbf{i}_s\in\mathcal{I}_s}(1-\alpha)^{-1}\bigoplus_{l=1}^{s}\mathcal{A}_{\mathbf{j}_{s-l}(\mathbf{i}_s)}\mathbb{W}\right)\\
&= \operatorname{co}\left(\bigcup_{\mathbf{i}_{s-1}\in\mathcal{I}_{s-1}}(1-\alpha)^{-1}\bigoplus_{l=0}^{s-1}\mathcal{A}_{\mathbf{j}_{s-1-l}(\mathbf{i}_{s-1})}\mathbb{W}\right)\\
&= D(s,\alpha)
\end{aligned}$$

which establishes the RPI property of the set $D(s,\alpha)$ and completes our proof.

APPENDIX B11 – Proof of Theorem 5

The proofs of Theorems 5 and 6 follow closely the proofs of Theorem 2 and 3 in [25] and are provided here for sake of completeness.

We first recall that if Φ is a convex, non–empty, compact set that contains the origin in its interior, for all $\alpha \in [0, 1)$ it holds that $\mathcal{H}(\Phi, (1-\alpha)^{-1}\Phi) \le \alpha(1-\alpha)^{-1}M$ where $M = \sup_{z\in\Phi} ||z||_p$ is finite [22]. For $s \in \mathbb{N}$, let $M(s) \triangleq \sup_{z\in D_s} ||z||_p$.

We proceed to prove (i). We have that $\mathcal{H}(D_s, (1-\alpha^0(s))^{-1}D_s) \le \alpha^0(s)(1-\alpha^0(s))^{-1}M(s)$. Moreover, since $D_s \subseteq D_\infty \subseteq (1-\alpha^0(s))^{-1}D_s$, $\mathcal{H}(D_\infty, (1-\alpha^0(s))^{-1}D_s) \le \alpha^o(s)(1-\alpha^o(s))^{-1}M(s)$. Assumption 2 and the fact that $M(s) \le M_\infty, \forall s \in \mathbb{N}$ and $M_\infty \triangleq \sup_{z\in D_\infty} ||z||_p$ is finite, yields that $\alpha^0(s) \to 0$ as $s \to \infty$. Thus, $\mathcal{H}(D_\infty, (1-\alpha^o(s))^{-1}D_s) \to 0$ as $s \to \infty$ and hence $D(s, \alpha^o(s)) = (1-\alpha^o(s))^{-1}D_s \to D_\infty$ in the Hausdorff metric.

The proof of (ii) follows similar arguments. We have that $\mathcal{H}(D_{s^0(\alpha)}, (1-\alpha)^{-1}D_{s^0(\alpha)}) \le \alpha(1-\alpha)^{-1}M(s^0(\alpha))$ and $\mathcal{H}(D_\infty, (1-\alpha)^{-1}D_{s^0(\alpha)}) \le \alpha(1-\alpha)^{-1}M(s^0(\alpha))$. Note that for $\alpha \searrow 0$ we have $s^0(\alpha) \to \infty$. Hence, $\mathcal{H}(D_\infty, (1-\alpha)^{-1}D_{s^0(\alpha)}) \to 0$ as $\alpha \searrow 0$ and therefore $D(s^0(\alpha), \alpha) \to D_\infty$ as $\alpha \searrow 0$.

APPENDIX B12 – Proof of Theorem 6

We refer to the proof of Theorem 5 for the definition of M_∞. Let $\varepsilon > 0$ and recall that $0 < M_\infty < \infty$ and $D_s \subseteq D_\infty$ for all $s \in \mathbb{N}$. Since D_s and D_∞ are convex and contain the origin, it follows that $\alpha(1-\alpha)^{-1}D_s \subseteq \alpha(1-\alpha)^{-1}D_\infty$ for any $s \in \mathbb{N}$ and $\alpha \in [0, 1)$. Note that the inclusion $\alpha(1-\alpha)^{-1}D_\infty \subseteq \mathbb{B}_p^n(\varepsilon)$ is true if $\alpha(1-\alpha)^{-1}M_\infty \le \varepsilon$ or, equivalently, if $\alpha \le \varepsilon(\varepsilon + M_\infty)^{-1}$. Hence, (56) is true for any $s \in \mathbb{N}$ and $\alpha \in [0, \bar{\alpha}]$, where $\bar{\alpha} \triangleq \varepsilon(\varepsilon + M_\infty)^{-1} \in [0, 1)$. Clearly, (48) is also true if we choose $\alpha \in (0, \bar{\alpha}]$ and $s = s^o(\alpha)$. This establishes the existence of a suitable pair (α, s) such that (48) and (56) hold simultaneously.

Let (s, α) be such that (48) and (56) are true. Since $D(s, \alpha) = (1-\alpha)^{-1}D_s$ is a convex and compact set that contains the origin, $D(s, \alpha) = (1-\alpha)^{-1}D_s = (1+\alpha(1-\alpha)^{-1})D_s = D_s \oplus \alpha(1-\alpha)^{-1}D_s$. Since $D_s \subseteq D_\infty \subseteq D(s, \alpha) \subseteq D_s \oplus \mathbb{B}_p^n(\varepsilon) \subseteq D_\infty \oplus \mathbb{B}_p^n(\varepsilon)$, it follows that $D(s, \alpha)$ is an RPI, outer ε-approximation of the mCRPI set D_∞.

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