

Efficient Robust Optimization for Robust Control with Constraints

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Abstract

This paper proposes an efficient computational technique for the optimal control of linear discrete-time systems subject to bounded disturbances with mixed polytopic constraints on the states and inputs. The problem of computing an optimal state feedback control policy, given the current state, is non-convex. A recent breakthrough has been the application of robust optimization techniques to reparameterise this problem as a convex program. While the reparameterised problem is theoretically tractable, the number of variables is quadratic in the number of stages or horizon length N and has no apparent exploitable structure, leading to computational time of $O(N^6)$ per iteration of an interior-point method. We focus on the case when the disturbance set is ∞ -norm bounded or the linear map of a hypercube, and the cost function involves the minimization of a quadratic cost. Here we make use of state variables to regain a sparse problem structure that is related to the structure of the original problem, that is, the policy optimization problem may be decomposed into a set of coupled finite horizon control problems. This decomposition can then be formulated as a highly structured quadratic program, solvable by primal-dual interior-point methods in which each iteration requires $O(N^3)$ time. This cubic iteration time can be guaranteed using a Riccati-based block factorization technique, which is standard in discrete-time optimal control. Numerical results are presented, using a standard sparse primal-dual interior point solver, which illustrate the efficiency of this approach.

Keywords: Constrained control, robust optimization, optimal control, robust control, receding horizon control, predictive control.

1 Introduction

Robust and predictive control

This paper is concerned with the efficient computation of optimal control policies for constrained discrete-time linear systems subject to bounded disturbances on the state. In particular, we consider the problem of finding, over a finite horizon of length N , a *feedback policy*

$$\pi := \{\mu_0(\cdot), \dots, \mu_{N-1}(\cdot)\} \quad (1)$$

for a discrete-time linear dynamical system of the form

$$x_{i+1} = Ax_i + Bu_i + w_i \quad (2)$$

$$u_i = \mu_i(x_0, \dots, x_i) \quad (3)$$

which guarantees satisfaction of a set of mixed constraints on the states and inputs at each time, for all possible realizations of the disturbances w_i , while minimizing a given cost function.

The states x_i and inputs u_i are constrained to lie in a compact and convex set \mathcal{Z} , i.e.

$$(x_i, u_i) \in \mathcal{Z}, \quad \forall i \in \{0, 1, \dots, N-1\} \quad (4)$$

with an additional terminal constraint $x_N \in X_f$. We assume nothing about the disturbances but that they lie in a given compact set W .

The above, rather abstract problem is motivated by the fact that for many real-life control applications, optimal operation nearly always occurs on or close to some constraints [36]. These constraints typically arise, for example, due to actuator limitations, safe regions of operation, or performance specifications. For safety-critical applications, in particular, it is crucial that some or all of these constraints are met, despite the presence of unknown disturbances.

Because of its importance, the above problem and derivations of it have been studied for some time now, with a large body of literature that falls under the broad banner of “robust control” (see [5, 49] for some seminal work on the subject). The field of *linear* robust control, which is mainly motivated by frequency-domain performance criteria [52] and does *not* explicitly consider time-domain constraints as in the above problem formulation, is considered to be mature and a number of excellent references are available on the subject [16, 23, 53]. In contrast, there are few tractable, non-conservative solutions to the above, constrained problem, even if all the constraint sets are considered to be polytopes or ellipsoids; see, for example, the literature on set invariance theory [7] or ℓ_1 optimal control [12, 17, 44, 46].

A control design method that is particularly suitable for the synthesis of controllers for systems with constraints, is predictive control [10, 36]. Predictive control is a family of optimal control techniques where, at each time instant, a finite-horizon constrained optimal control problem is solved using tools from mathematical programming. The solution to this optimization problem is usually implemented in a receding horizon fashion, i.e. at each time instant, a measurement of the system is obtained, the associated optimization problem is solved and only the first control input in the optimal policy is implemented. Because of this ability to solve a sequence of complicated, *constrained* optimal control problems in real-time, predictive control is synonymous with “advanced control” in the chemical process industries [40].

The theory on predictive control *without* disturbances is relatively mature and most of the fundamental problems are well-understood. However, despite recent advances, there are many open questions remaining in the area of *robust* predictive control [3, 37, 38]. In particular, efficient optimization methods have to be developed for solving the above problem before robust predictive control methods can be applied to unstable or safety-critical applications in areas such as aerospace and automotive applications [45].

Robust control models

The core difficulty with the problem (1)–(4) is that optimizing the feedback policy π over *arbitrary* nonlinear functions is extremely difficult, in general. Proposals which take this approach, such as [43], are intractable for all but the smallest problems, since they generally require enumeration of all possible disturbance realizations generated from the set W . Conversely, optimization over *open-loop* control sequences, while tractable, is considered unacceptable since problems of infeasibility or instability may easily arise [38].

An obvious sub-optimal proposal is to parameterize the control policy π in terms of *affine* functions of the sequence of states, i.e. to parameterize the control sequence as

$$u_i = \sum_{j=0}^i L_{i,j} x_j + g_i \quad (5)$$

where the matrices $L_{i,j}$ and vectors g_i are decision variables. However, the set of constraint admissible policies of this form is easily shown to be *non-convex* in general. As a result, most proposals that take this approach [2, 11, 29, 32, 39] fix a stabilising feedback gain K , then parameterize the control sequence as $u_i = Kx_i + g_i$ and optimize the design parameters g_i . This approach, though tractable, is problematic, as it is unclear how one should select the gain K to minimize conservativeness.

A recent breakthrough [22] showed that the problem of optimizing over state feedback policies of the form (5) is equivalent to a *convex* optimization problem using disturbance feedback policies of the form

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i. \quad (6)$$

Robust optimization modelling techniques [4, 24] are used to eliminate the unknown disturbances w_j and formulate the admissible set of control policies with $O(N^2 mn)$ variables, where N is the horizon length as above, and m and n are the respective dimensions of the controls u_i and states x_i at each stage. This implies that, given a suitable objective function, an optimal affine state feedback policy (5) can be found in time that is polynomially bounded in the size of the problem data.

Efficient computational in robust optimal control

In the present paper we demonstrate that an optimal policy of the form (6), equivalently (5), can be efficiently calculated in practise, given suitable polytopic assumptions on the constraint sets W , \mathcal{Z} and X_f . This result is critical for practical applications, since one would generally implement a controller in a receding horizon fashion by calculating, on-line and at each time instant, an admissible control policy (5), given the current state x . Such a control strategy has been shown to allow for the synthesis of stabilizing, nonlinear time-invariant control laws that guarantee satisfaction of the constraints \mathcal{Z} for all time, for all possible disturbance sequences generated from W [22].

While convexity of the robust optimal problem arising out of (6) is key, the resulting optimization problem is a dense convex quadratic program with $O(N^2)$ variables (see Section 4, cf. [22]), assuming N dominates the dimension of controls m and states n at each stage. Hence each iteration of an interior-point method will require solving a dense linear system and thus require $O(N^6)$ time. This situation is common, for example, in the rapidly growing number of aerospace and automotive applications of predictive control [36, Sec. 3.3] [40]. We show that when the disturbance set is ∞ -norm bounded or the linear map of a hypercube, the special structure of the robust optimal control problem can be exploited to devise a sparse formulation of the problem, thereby realizing a substantial reduction in computational effort to $O(N^3)$ work per interior-point iteration.

We demonstrate that the cubic-time performance of interior-point algorithms at each step can be guaranteed when using a factorization technique based on Riccati recursion and block elimination. Numerical results are presented that demonstrate that the technique is computationally feasible for systems of appreciable complexity using the standard sparse linear system solver MA27 [26] within the primal-dual interior-point solver OOQP [19]. We compare this primal-dual interior-point approach to the sparse active-set method PATH [14] on both the dense and sparse problem formulations. Our results suggest that the interior-point method applied to the sparse formulation is the most practical method for solving robust optimal control problems, at least in the “cold start” situation when optimal active set information is unavailable.

A final remark is that the sparse formulation of robust optimal control results from a decomposition technique that can be used to separate the problem into a set of coupled finite horizon control problems. This reduction of effort is the analogue, for robust control, to the situation in classical unconstrained optimal control in which Linear Quadratic Regulator (LQR) problems can be solved in $O(N)$ time, using a Riccati [1, Sec. 2.4] or Differential Dynamic Programming [27] technique in which the state feedback equation $x^+ = Ax + Bu$ is explicit in every stage, compared to $O(N^3)$ time for the more compact formulation in which states are eliminated from the system. More direct motivation for our work comes from [6, 13, 41, 50], which describe efficient implementations of optimization methods for solving optimal control problems with state and control constraints, though without disturbances.

Contents

The paper is organized as follows: Sections 2 and 3 introduce the optimal control problem considered throughout the paper. Sections 4 and 5 describe the equivalent affine disturbance feedback policy employed to solve the optimal control problem, as well as an equivalent formulation which can be decomposed into a highly structured, singly-bordered block-diagonal quadratic program through reintroduction of appropriate state variables. Section 6 demonstrates that, when using a primal-dual interior-point solution technique, the decomposed quadratic program can always be solved in time cubic in the horizon length at each interior-point iteration. Section 7 demonstrates through numerical examples that the proposed decomposition can be solved much more efficiently than the equivalent original formulation, and the paper concludes in Section 8 with suggestions for further research.

2 Definitions and Standing Assumptions

Consider the following discrete-time linear time-invariant system:

$$x^+ = Ax + Bu + w, \quad (7)$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. It is further assumed that the current and future values of the disturbance are unknown and may change unpredictably from one time instant to the next, but are contained in a convex and compact (closed and bounded) set W , which contains the origin. Initially, we make no further assumptions on the set W .

The system is subject to mixed constraints on the state and input:

$$\mathcal{Z} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \quad (8)$$

where the matrices $C \in \mathbb{R}^{s \times n}$, $D \in \mathbb{R}^{s \times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints that define \mathcal{Z} . It is assumed that \mathcal{Z} is bounded and contains the origin in its interior.

A design goal is to guarantee that the state and input of the closed-loop system remain in \mathcal{Z} for all time and for all allowable disturbance sequences.

In addition to \mathcal{Z} , a target/terminal constraint set X_f is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \quad (9)$$

where the matrix $Y \in \mathbb{R}^{r \times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f . It is assumed that X_f is bounded and contains the origin in its interior. The set X_f can, for example, be used as a target set in time-optimal control or, if defined to be robust positively invariant, to design a receding horizon controller with guaranteed invariance and stability properties [22].

Before proceeding, we define some additional notation. In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{nN}$, respectively, as

$$\begin{aligned} \mathbf{x} &:= \text{vec}(x_0, \dots, x_{N-1}, x_N), \\ \mathbf{u} &:= \text{vec}(u_0, \dots, u_{N-1}), \\ \mathbf{w} &:= \text{vec}(w_0, \dots, w_{N-1}), \end{aligned}$$

where $x_0 = x$ denotes the current measured value of the state and $x_{i+1} := Ax_i + Bu_i + w_i$, $i = 0, \dots, N-1$ denotes the prediction of the state after i time instants into the future. Finally, let the set $\mathcal{W} := W^N := W \times \dots \times W$, so that $\mathbf{w} \in \mathcal{W}$.

3 An Affine State Feedback Parameterization

One natural approach to controlling the system in (7), while ensuring the satisfaction of the constraints (8)–(9), is to search over the set of time-varying affine state feedback control policies. We thus consider policies of the form:

$$u_i = \sum_{j=0}^i L_{i,j} x_j + g_i, \quad \forall i \in \mathbb{Z}_{[0, N-1]}, \quad (11)$$

where each $L_{i,j} \in \mathbb{R}^{m \times n}$ and $g_i \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} := \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \quad \mathbf{g} := \begin{bmatrix} g_0 \\ \vdots \\ g_{N-1} \end{bmatrix}, \quad (12)$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}$. For a given initial state x (since the system is time-invariant, the current time can always be taken as zero), we say that the pair (\mathbf{L}, \mathbf{g}) is admissible if the control policy (11) guarantees that for all allowable disturbance sequences of length N , the constraints (8) are satisfied over the horizon $i = 0, \dots, N-1$ and that the state is in the target set (9) at the end of the horizon. More precisely, the set of admissible (\mathbf{L}, \mathbf{g}) is defined as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \left| \begin{array}{l} (\mathbf{L}, \mathbf{g}) \text{ satisfies (12), } x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^i L_{i,j} x_j + g_i \\ (x_i, u_i) \in \mathcal{Z}, x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0, N-1]}, \forall \mathbf{w} \in \mathcal{W} \end{array} \right. \right\} \quad (13)$$

and the set of initial states x for which an admissible control policy of the form (11) exists is defined as

$$X_N^{sf} := \left\{ x \in \mathbb{R}^n \mid \Pi_N^{sf}(x) \neq \emptyset \right\}. \quad (14)$$

It is critical to note that, in general, it is *not possible* to select a single pair (\mathbf{L}, \mathbf{g}) such that $(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)$ for all $x \in X_N^{sf}$. Indeed, it is possible that for some pair $(x, \tilde{x}) \in X_N^{sf} \times X_N^{sf}$, $\Pi_N^{sf}(x) \cap \Pi_N^{sf}(\tilde{x}) = \emptyset$.

For problems of non-trivial size, it is therefore necessary to calculate an admissible pair (\mathbf{L}, \mathbf{g}) on-line, given a measurement of the current state x , rather than fixing (\mathbf{L}, \mathbf{g}) off-line. Once an admissible control policy is computed for the current state, it can be implemented either in a classical time-varying, time-optimal or receding-horizon fashion.

In particular, we define an optimal policy pair $(\mathbf{L}^*(x), \mathbf{g}^*(x)) \in \Pi_N^{sf}(x)$ to be one which minimizes the value of a cost function that is quadratic in the disturbance-free state and input sequence. We thus define:

$$V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{w}) := \sum_{i=0}^{N-1} \frac{1}{2} (\|\tilde{x}_i\|_Q^2 + \|\tilde{u}_i\|_R^2) + \frac{1}{2} \|\tilde{x}_N\|_P^2 \quad (15)$$

where $\tilde{x}_0 = x$, $\tilde{x}_{i+1} = A\tilde{x}_i + B\tilde{u}_i + w_i$ and $\tilde{u}_i = \sum_{j=0}^i L_{i,j} \tilde{x}_j + g_i$ for $i = 0, \dots, N-1$; the matrices Q and P are positive semidefinite, and R is positive definite. We define an optimal policy pair as

$$(\mathbf{L}^*(x), \mathbf{g}^*(x)) := \underset{(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)}{\operatorname{argmin}} V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{0}). \quad (16)$$

Before proceeding, we also define the value function $V_N^* : X_N^{sf} \rightarrow \mathbb{R}_{\geq 0}$ to be

$$V_N^*(x) := \min_{(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)} V_N(x, \mathbf{L}, \mathbf{g}, 0). \quad (17)$$

For the receding-horizon control case, a *time-invariant* control law $\mu_N : X_N^{sf} \rightarrow \mathbb{R}^m$ can be implemented by using the first part of this optimal control policy at each time instant, i.e.

$$\mu_N(x) := L_{0,0}^*(x)x + g_0^*(x). \quad (18)$$

We emphasize that, due to the dependence of (13) on the current state x , the control law $\mu_N(\cdot)$ will, in general, be a *nonlinear* function with respect to the current state, even though it may have been defined in terms of the class of affine state feedback policies (11).

Remark 1. Note that the state feedback policy (11) includes the well-known class of “pre-stabilizing” control policies [11, 29, 32, 39], in which the control policy takes the form $u_i = Kx_i + c_i$, where K is computed off-line and only c_i is computed on-line.

The control law $\mu_N(\cdot)$ has many attractive geometric and system-theoretic properties. In particular, implementation of the control law $\mu_N(\cdot)$ renders the set X_N^{sf} robust positively invariant, i.e. if $x \in X_N^{sf}$, then it can be shown that $Ax + B\mu_N(x) + w \in X_N^{sf}$ for all $w \in W$, subject to certain technical conditions on the terminal set X_f . Furthermore, the closed-loop system is guaranteed to be input-to-state (ISS) stable when W is a polytope, under suitable assumptions on Q , P , R , and X_f . The reader is referred to [22] for a proof of these results and a review of other system-theoretic properties of this parameterization.

Unfortunately, such a control policy is seemingly very difficult to compute, since the set $\Pi_N^{sf}(x)$ and cost function $V_N(x, \cdot, \cdot, 0)$ are non-convex; however, it is possible to convert this non-convex optimization problem to an equivalent *convex* problem through an appropriate reparameterization. This parameterization is introduced in the following section.

4 Affine Disturbance Feedback Control Policies

An alternative to (11) is to parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, \quad \forall i \in \mathbb{Z}_{[0, N-1]}, \quad (19)$$

where each $M_{i,j} \in \mathbb{R}^{m \times n}$ and $v_i \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance sequence is easily calculated as the difference between the predicted and actual states at each step, i.e.

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0, N-1]}. \quad (20)$$

The above parameterization appears to have originally been suggested some time ago within the context of stochastic programs with recourse [18]. More recently, it has been revisited as a means for finding solutions to a class of robust optimization problems, called affinely adjustable robust counterpart (AARC) problems [4, 24], and robust model predictive control problems [33, 34, 47, 48].

Define the variable $\mathbf{v} \in \mathbb{R}^{mN}$ and the block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1, N-2} & 0 \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} v_0 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad (21)$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}$. Define the set of admissible (\mathbf{M}, \mathbf{v}) , for which the constraints (8) and (9) are satisfied, as:

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (21), } x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \\ (x_i, u_i) \in \mathcal{Z}, x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0, N-1]}, \forall \mathbf{w} \in \mathcal{W} \end{array} \right. \right\}, \quad (22)$$

and define the set of initial states x for which an admissible control policy of the form (19) exists as

$$X_N^{df} := \{x \in \mathbb{R}^n \mid \Pi_N^{df}(x) \neq \emptyset\}. \quad (23)$$

As shown in [30], it is possible to eliminate the universal quantifier in (22) and construct matrices $F \in \mathbb{R}^{(sN+r) \times mN}$, $G \in \mathbb{R}^{(sN+r) \times nN}$ and $T \in \mathbb{R}^{(sN+r) \times n}$, and vector $c \in \mathbb{R}^{sN+r}$ (defined in Appendix A) such that the set of feasible pairs (\mathbf{M}, \mathbf{v}) can be written as:

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (21),} \\ F\mathbf{v} + \text{vec max}_{\mathbf{w} \in \mathcal{W}}(F\mathbf{M} + G)\mathbf{w} \leq c + Tx \end{array} \right. \right\}, \quad (24)$$

where $\text{vec max}_{\mathbf{w} \in \mathcal{W}}(F\mathbf{M} + G)\mathbf{w}$ denotes row-wise maximization, i.e. if $(F\mathbf{M} + G)_i$ denotes the i^{th} row of the matrix $F\mathbf{M} + G$, then

$$\text{vec max}_{\mathbf{w} \in \mathcal{W}}(F\mathbf{M} + G)\mathbf{w} := \text{vec} \left(\max_{\mathbf{w} \in \mathcal{W}}(F\mathbf{M} + G)_1 \mathbf{w}, \dots, \max_{\mathbf{w} \in \mathcal{W}}(F\mathbf{M} + G)_{sN+r} \mathbf{w} \right). \quad (25)$$

Note that these maxima always exist when the set \mathcal{W} is compact.

We are interested in this control policy parameterization primarily due to the following two properties, proof of which may be found in [22]:

Theorem 1 (Convexity). For a given state $x \in X_N^{df}$, the set of admissible affine disturbance feedback parameters $\Pi_N^{df}(x)$ is convex. Furthermore, the set of states X_N^{df} , for which at least one admissible affine disturbance feedback parameter exists, is also convex.

Theorem 2 (Equivalence). The set of admissible states $X_N^{df} = X_N^{sf}$. Additionally, given any $x \in X_N^{sf}$, for any admissible (\mathbf{L}, \mathbf{g}) an admissible (\mathbf{M}, \mathbf{v}) can be constructed that yields the same input and state sequence for all allowable disturbances, and vice-versa.

Together, these results enable efficient implementation of the control law $u = \mu_N(x)$ in (18) by replacing the non-convex optimization problem (16) with an equivalent convex problem. If we define the nominal states $\hat{x}_i \in \mathbb{R}^n$ to be the states when no disturbances occur, i.e. $\hat{x}_{i+1} = A\hat{x}_i + Bv_i$, and define $\hat{\mathbf{x}} \in \mathbb{R}^{nN}$ as

$$\hat{\mathbf{x}} := \text{vec}(x, \hat{x}_1, \dots, \hat{x}_N) = \mathbf{A}x + \mathbf{B}\mathbf{v}. \quad (26)$$

where $\mathbf{A} \in \mathbb{R}^{n(N+1) \times n}$ and $\mathbf{B} \in \mathbb{R}^{n(N+1) \times mN}$ are defined in Appendix A, then a quadratic cost function similar to that in (15) can be written as

$$V_N^{df}(x, \mathbf{v}) := \sum_{i=0}^{N-1} \frac{1}{2} (\|\hat{x}_i\|_{\mathcal{Q}}^2 + \|v_i\|_{\mathcal{R}}^2) + \frac{1}{2} \|\hat{x}_N\|_{\mathcal{P}}^2, \quad (27)$$

or, in vectorized form, as

$$V_N^{df}(x, \mathbf{v}) = \frac{1}{2} \|\mathbf{A}x + \mathbf{B}\mathbf{v}\|_{\mathcal{Q}}^2 + \frac{1}{2} \|\mathbf{v}\|_{\mathcal{R}}^2 \quad (28)$$

where $\mathcal{Q} := [I \otimes Q \quad P]$ and $\mathcal{R} := I \otimes R$. This cost can then be optimized over allowable policies in (24), forming a convex optimization problem in the variables \mathbf{M} and \mathbf{v} :

$$\min_{\mathbf{M}, \mathbf{v}} V_N^{df}(x, \mathbf{v}) \quad \text{s.t. } (\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x) \quad (29)$$

As a direct result of the equivalence of the two parameterizations, the minimum of $V_N^{df}(x, \cdot)$ evaluated over the admissible policies $\Pi_N^{df}(x)$ is equal to the minimum of $V_N(x, \cdot, \cdot, 0)$ in (16), i.e.

$$\min_{(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)} V_N^{df}(x, \mathbf{v}) = \min_{(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)} V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{0}) \quad (30)$$

The control law $\mu_N(\cdot)$ in (18) can then be implemented using the first part of the optimal $\mathbf{v}^*(\cdot)$ at each step, i.e.

$$\mu_N(x) = v_0^*(x) = L_{0,0}^*(x)x + g_0^*(x) \quad (31)$$

where

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) := \underset{(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)}{\text{argmin}} V_N^{df}(x, \mathbf{v}) \quad (32)$$

$$\mathbf{v}^*(x) =: \text{vec}(v_0^*(x), \dots, v_{N-1}^*(x)) \quad (33)$$

which requires the minimization of a *convex* function over a *convex* set. The receding horizon control law $\mu_N(\cdot)$ in (18) is thus practically realizable. Note that since the cost function (27) is strictly convex in \mathbf{v} , both the minimizer $\mathbf{v}^*(x)$ and the resulting control law (31) are uniquely defined for each x [42, Thm 2.6], and the value function $V_N^*(\cdot)$ is convex and continuous on the interior of X_N^{sf} [42, Thm. 2.35, Cor. 3.32].

Until this point, no constraints have been imposed on the disturbance set W other than the requirement that it be compact; Theorems 1 and 2 hold even for non-convex disturbance sets. In the remainder of this section, we consider the special cases where W is a polytopic or ∞ -norm bounded set.

4.1 Optimization Over Polytopic Disturbance Sets

We consider the special case where the constraint set describes a polytope (closed and bounded polyhedron). In this case the disturbance set may be written as

$$\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^{nN} \mid S\mathbf{w} \leq h\} \quad (34)$$

where $S \in \mathbb{R}^{t \times n}$ and $h \in \mathbb{R}^t$ (note that this includes cases where the disturbance set is time varying). Note that both 1- and ∞ -norm disturbance sets W can be characterized in this manner.

In this case, we can write the dual problem for each row of $\text{vec} \max_{\mathbf{w} \in \mathcal{W}} (F\mathbf{M} + G)\mathbf{w}$, and solve the convex control policy optimization problem (29) as a single quadratic program.

4.1.1 Introduction of Dual Variables to the Robust Problem

With a slight abuse of notation, we recall that for a general LP in the form

$$\min_z c^T z, \quad \text{s.t. } Az = b, z \geq 0 \quad (35)$$

the same problem can be solved in dual form by solving the dual problem

$$\max_w b^T w, \quad \text{s.t. } A^T w \leq c \quad (36)$$

where, for each feasible pair (z, w) , it is always true that $b^T w \leq c^T z$. Using this idea, if we define the i^{th} row of $(F\mathbf{M} + G)$ as $(F\mathbf{M} + G)_i$, then the dual of each row of (25) can be written as

$$\max_{\mathbf{w} \in \mathcal{W}} (F\mathbf{M} + G)_i \mathbf{w} = \min_{\mathbf{z}_i} h^T \mathbf{z}_i, \quad \text{s.t. } S^T \mathbf{z}_i = (F\mathbf{M} + G)_i^T, \mathbf{z}_i \geq 0, \quad (37)$$

where the vectors $\mathbf{z}_i \in \mathbb{R}^t$ represents the dual variables associated with the i^{th} row. By combining these dual variables into a matrix

$$\mathbf{Z} := [\mathbf{z}_1 \quad \dots \quad \mathbf{z}_N] \quad (38)$$

the set $\Pi_N^{\text{df}}(x)$ can be written in terms of purely linear constraints:

$$\Pi_N^{\text{df}}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (21), } \exists \mathbf{Z} \text{ s.t.} \\ F\mathbf{v} + \mathbf{Z}^T h \leq c + Tx \\ F\mathbf{M} + G = \mathbf{Z}^T S, \mathbf{Z} \geq 0 \end{array} \right. \right\}, \quad (39)$$

where all inequalities are element-wise.

Note that the convex optimization problem (29) now requires optimization over the polytopic set (39), leading to a quadratic programming problem in the variables \mathbf{M} , \mathbf{Z} and \mathbf{v} .

Remark 2. *When the disturbance set is polytopic, it can be shown that the value function $V_N^*(\cdot)$ is piecewise quadratic on X_N , and the resulting control law $\mu_N(\cdot)$ is piecewise affine [22].*

4.2 Optimization Over ∞ -Norm Bounded Disturbance Sets

In the remainder of this paper, we consider the particular case where W is generated as the linear map of a hypercube. Define

$$W = \{w \in \mathbb{R}^n \mid w = Ed, \|d\|_\infty \leq 1\}, \quad (40)$$

where $E \in \mathbb{R}^{n \times l}$, so that the stacked generating disturbance sequence $\mathbf{d} \in \mathbb{R}^{lN}$ is

$$\mathbf{d} = \text{vec}(d_0, \dots, d_{N-1}), \quad (41)$$

and define the matrix $J := I_N \otimes E \in \mathbb{R}^{Nn \times Nl}$, so that $\mathbf{w} = J\mathbf{d}$. As shown in [30], an analytical solution to the row-wise maximization in (24) is easily found. From the definition of the dual norm [25], when the generating disturbance d is any p -norm bounded signal given as in (40), then

$$\max_{w \in W} a^T w = \|E^T a\|_q \quad (42)$$

for any vector $a \in \mathbb{R}^n$, where $1/p + 1/q = 1$.

Straightforward application of (42) to the row wise-maximization in (24) (with $q = 1$) yields

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (21)} \\ F\mathbf{v} + \text{abs}(F\mathbf{M}J + GJ)\mathbf{1} \leq c + Tx \end{array} \right. \right\}, \quad (43)$$

where $\text{abs}(F\mathbf{M}J + GJ)\mathbf{1}$ is a vector formed from the 1-norms of the rows of $(F\mathbf{M}J + GJ)$. This can be written as a set of purely affine constraints by introducing slack variables and rewriting as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (21), } \exists \mathbf{\Lambda} \text{ s.t.} \\ F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} \leq c + Tx \\ -\mathbf{\Lambda} \leq (F\mathbf{M}J + GJ) \leq \mathbf{\Lambda} \end{array} \right. \right\}. \quad (44)$$

As in the case for general polytopic disturbance sets, the cost function (27) can then be minimized over allowable policies in (44) by forming a quadratic program in the variables \mathbf{M} , $\mathbf{\Lambda}$, and \mathbf{v} , i.e.

$$\min_{\mathbf{M}, \mathbf{\Lambda}, \mathbf{v}} \frac{1}{2} \|\mathbf{A}x + \mathbf{B}\mathbf{v}\|_{\mathcal{Q}}^2 + \frac{1}{2} \|\mathbf{v}\|_{\mathcal{R}}^2 \quad (45a)$$

subject to:

$$F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} \leq c + Tx \quad (45b)$$

$$-\mathbf{\Lambda} \leq (F\mathbf{M}J + GJ) \leq \mathbf{\Lambda}. \quad (45c)$$

Remark 3. *The total number of decisions variables in (45) is mN in \mathbf{v} , mnN^2 in \mathbf{M} , and $(slN^2 + rln)$ in $\mathbf{\Lambda}$, with a number of constraints equal to $(sN + r) + 2(slN^2 + rln)$, or $\mathcal{O}(N^2)$ overall. For a naive interior-point computational approach using a dense factorization method, the resulting quadratic program would thus require computation time of $\mathcal{O}(N^6)$ at each iteration.*

4.2.1 Writing $\Pi_N^{df}(x)$ in Separable Form

Next, define the following variable transformation:

$$\mathbf{U} := \mathbf{M}J \quad (46)$$

such that the matrix $\mathbf{U} \in \mathbb{R}^{mN \times lN}$ has block lower triangular structure similar to that defined in (21) for \mathbf{M} , i.e.

$$\mathbf{U} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ U_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ U_{N-1,0} & \cdots & U_{N-1,N-2} & 0 \end{bmatrix}. \quad (47)$$

We note that this parameterization is tantamount to parameterizing the control policy directly in terms of the generating disturbances d_i , so that $u_i = \sum_{j=0}^{i-1} U_{i,j}d_j + v_i$, or $\mathbf{u} = \mathbf{U}\mathbf{d} + \mathbf{v}$. These generating disturbances are obviously not directly measurable, and must instead be inferred from the real disturbances w_i .

Proposition 1. *For each $w \in W$, a generating d such that $\|d\|_{\infty} \leq 1$ and $w = Ed$ can always be found. If E is full column rank, the generating disturbance is unique¹.*

¹Note that if E is *not* full column rank, an admissible d can still always be chosen.

We define the set of admissible (\mathbf{U}, \mathbf{v}) as

$$\Pi_N^{uf}(x) := \left\{ (\mathbf{U}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{U}) \text{ satisfies (47), } \exists \mathbf{\Lambda} \text{ s.t.} \\ F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} \leq c + Tx \\ -\mathbf{\Lambda} \leq (F\mathbf{U} + GJ) \leq \mathbf{\Lambda} \end{array} \right. \right\}, \quad (48)$$

and the set of states for which such a controller exists as

$$X_N^{uf} := \{x \in \mathbb{R}^n \mid \Pi_N^{uf}(x) \neq \emptyset\}. \quad (49)$$

Proposition 2. $X_N^{df} \subseteq X_N^{uf}$. If E is full column rank, then $X_N^{df} = X_N^{uf} = X_N^{sf}$. Additionally, given any $x \in X_N^{sf}$, for any admissible pair (\mathbf{U}, \mathbf{v}) an admissible (\mathbf{L}, \mathbf{g}) can be constructed that yields the same input and state sequence for all allowable disturbances, and vice-versa.

Proof. It is easy to verify that $X_N^{df} \subseteq X_N^{uf}$, by noting that if $x \in X_N^{df}$ with admissible control policy (\mathbf{M}, \mathbf{v}) , then, from (44), the control sequence \mathbf{u} can be written as $\mathbf{u} = \mathbf{M}J\mathbf{d} + \mathbf{v}$, and the constraints in (48) satisfied by selecting $\mathbf{U} = \mathbf{M}J$ so that $x \in X_N^{df} \Rightarrow x \in X_N^{uf}$.

If E has full column rank, then $J = I_N \otimes E$ also has full column rank, and therefore has a left inverse J^\dagger satisfying $J^\dagger J = I$. If $x \in X_N^{uf}$, then there exists an admissible policy (\mathbf{U}, \mathbf{v}) that results in a control sequence $\mathbf{u} = \mathbf{U}\mathbf{d} + \mathbf{v}$, and the constraints in (44) are satisfied by selecting $\mathbf{M} = \mathbf{U}J^\dagger$, so that $x \in X_N^{uf} \Rightarrow x \in X_N^{df}$. The remainder of the proof follows directly from Theorem 2. The reader is referred to [22] for a method of constructing such an (\mathbf{L}, \mathbf{g}) given an admissible (\mathbf{M}, \mathbf{v}) . \square

Optimization of the cost function (27) over the set (48) thus yields a quadratic program in the variables \mathbf{U} , $\mathbf{\Lambda}$ and \mathbf{v} :

$$\min_{\mathbf{U}, \mathbf{\Lambda}, \mathbf{v}} \frac{1}{2} \|\mathbf{A}x + \mathbf{B}\mathbf{v}\|_{\mathcal{Q}}^2 + \frac{1}{2} \|\mathbf{v}\|_{\mathcal{R}}^2 \quad (50a)$$

subject to:

$$F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} \leq c + Tx \quad (50b)$$

$$-\mathbf{\Lambda} \leq (F\mathbf{U} + GJ) \leq \mathbf{\Lambda}. \quad (50c)$$

Remark 4. The critical feature of the quadratic program (50) is that the columns of the variables \mathbf{U} and $\mathbf{\Lambda}$ are decoupled in the constraint (50c). This allows column-wise separation of the constraint into a number of subproblems, subject to the coupling constraint (50b).

5 Recovering Structure in the Robust Control Problem

The quadratic program (QP) defined in (50) can be rewritten in a more computationally attractive form by re-introducing the eliminated state variables to achieve greater structure. The re-modelling process separates the original problem into subproblems; a nominal problem, consisting of that part of the state resulting from the nominal control vector \mathbf{v} , and a set of perturbation problems, each representing those components of the state resulting from each of the columns of (50c) in turn.

Nominal States and Inputs

We first define a constraint contraction vector $\delta\mathbf{c} \in \mathbb{R}^{sN+r}$ such that

$$\delta\mathbf{c} := \text{vec}(\delta c_0, \dots, \delta c_N) = \mathbf{\Lambda}\mathbf{1}, \quad (51)$$

so that the constraint (50b) becomes

$$F\mathbf{v} + \delta\mathbf{c} \leq c + T\mathbf{x}. \quad (52)$$

Recalling that the nominal states \hat{x}_i are defined in (26) as the expected states given no disturbances, it is easy to show that the constraint (52) can be written explicitly in terms of the nominal controls v_i and states \hat{x}_i as

$$\hat{x}_0 = x, \quad (53a)$$

$$\hat{x}_{i+1} - A\hat{x}_i - Bv_i = 0, \quad \forall i \in \mathbb{Z}_{[0, N-1]} \quad (53b)$$

$$C\hat{x}_i + Dv_i + \delta c_i \leq b, \quad \forall i \in \mathbb{Z}_{[0, N-1]} \quad (53c)$$

$$Y\hat{x}_N + \delta c_N \leq z, \quad (53d)$$

which is in a form that is *exactly the same* as that in conventional receding horizon control problem with no disturbances, but with the right-hand-sides of the state and input constraints at each stage i modified by δc_i ; compare (53b)–(53d) and (7)–(9) respectively.

Perturbed States and Inputs

We next consider the effects of each of the columns of $(F\mathbf{U} + GJ)$ in turn, and seek to construct a set of problems similar to that in (53). We treat each column as the output of a system subject to a unit impulse in a single element of \mathbf{d} , and construct a subproblem that calculates the effect of that disturbance on the nominal problem constraints (53c)–(53d) by determining its contribution to the total constraint contraction vector $\delta\mathbf{c}$.

From the original QP constraint (50c), the constraint contraction vector $\delta\mathbf{c}$ can be written as

$$\text{abs}(F\mathbf{U} + GJ)\mathbf{1} \leq \Lambda\mathbf{1} = \delta\mathbf{c}, \quad (54)$$

the left hand side of which can be rewritten as

$$\text{abs}(F\mathbf{U} + GJ)\mathbf{1} = \sum_{p=1}^{lN} \text{abs}((F\mathbf{U} + GJ)e_p). \quad (55)$$

Define $\mathbf{y}^p \in \mathbb{R}^{sN+r}$ and $\delta\mathbf{c}^p \in \mathbb{R}^{sN+r}$ as

$$\mathbf{y}^p := (F\mathbf{U} + GJ)e_p \quad (56)$$

$$\delta\mathbf{c}^p := \text{abs}(\mathbf{y}^p). \quad (57)$$

The unit vector e_p represents a unit disturbance in some element j of the generating disturbance d_k at some time step k , with no disturbances at any other step². If we denote the j^{th} column of E as $E_{(j)}$, then it is easy to recognize \mathbf{y}^p as the stacked output vector of the system

$$(u_i^p, x_i^p, y_i^p) = 0, \quad \forall i \in \mathbb{Z}_{[0, k]} \quad (58a)$$

$$x_{k+1}^p = E_{(j)}, \quad (58b)$$

$$x_{i+1}^p - Ax_i^p - Bu_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1, N-1]} \quad (58c)$$

$$y_i^p - Cx_i^p - Du_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1, N-1]} \quad (58d)$$

$$y_N^p - Yx_N^p = 0, \quad (58e)$$

where $\mathbf{y}^p = \text{vec}(y_0^p, \dots, y_N^p)$. The inputs u_i^p of this system come directly from the p^{th} column of the matrix \mathbf{U} , and thus represent columns of the sub-matrices $U_{i,k}$. If the constraint terms $\delta\mathbf{c}^p$

²Note that this implies $p = lk + j$, $k = (p - j)/l$ and $j = 1 + (p - 1) \bmod l$.

for each subproblem are similarly written as $\delta c^p = \text{vec}(\delta c_0^p, \dots, \delta c_N^p)$, then each component must satisfy $\delta c_i^p = \text{abs}(y_i^p)$, or in linear inequality constraint form

$$-\delta c_i^p \leq y_i^p \leq \delta c_i^p. \quad (59)$$

It is of course important to note that the terminal output and constraint terms are not of the same dimension as the other terms in general, i.e. $y_i, \delta c_i \in \mathbb{R}^s \forall i \in \mathbb{Z}_{[0, N-1]}$, and $y_N, \delta c_N \in \mathbb{R}^r$. Note also that for the p^{th} subproblem, representing a disturbance at stage k , the constraint contraction terms are zero prior to stage $(k+1)$.

By further defining

$$\bar{C} := \begin{bmatrix} +C \\ -C \end{bmatrix} \quad \bar{D} := \begin{bmatrix} +D \\ -D \end{bmatrix} \quad \bar{Y} := \begin{bmatrix} +Y \\ -Y \end{bmatrix} \quad H := \begin{bmatrix} -I_s \\ -I_s \end{bmatrix} \quad H_f := \begin{bmatrix} -I_r \\ -I_r \end{bmatrix}, \quad (60)$$

equations (58d) and (58e) can be combined with (59) to give

$$\bar{C}x_i^p + \bar{D}u_i^p + H\delta c_i^p \leq 0, \quad \forall i \in \mathbb{Z}_{[k+1, N-1]} \quad (61a)$$

$$\bar{Y}x_N^p + H_f\delta c_N^p \leq 0. \quad (61b)$$

5.1 Complete Robust Control Problem

We can now restate the complete robust optimization problem (50) as:

$$\min_{\substack{\hat{x}_1, \dots, \hat{x}_N, v_0, \dots, v_{N-1}, \delta c_0, \dots, \delta c_N, \\ x_0^1, \dots, x_N^1, u_0^1, \dots, u_{N-1}^1, \delta c_0^1, \dots, \delta c_N^1, \\ \dots \\ x_0^{lN}, \dots, x_N^{lN}, u_0^{lN}, \dots, u_{N-1}^{lN}, \delta c_0^{lN}, \dots, \delta c_N^{lN}}} \sum_{i=0}^{N-1} \left(\frac{1}{2} \|\hat{x}_i\|_Q^2 + \frac{1}{2} \|v_i\|_R^2 \right) + \frac{1}{2} \|\hat{x}_N\|_P^2 \quad (62)$$

subject to (53), (58a)–(58c) and (61), which we restate here for convenience:

$$\hat{x}_0 = x, \quad (63a)$$

$$\hat{x}_{i+1} - A\hat{x}_i - Bv_i = 0, \quad \forall i \in \mathbb{Z}_{[0, N-1]} \quad (63b)$$

$$C\hat{x}_i + Dv_i + \delta c_i \leq b, \quad \forall i \in \mathbb{Z}_{[0, N-1]} \quad (63c)$$

$$Y\hat{x}_N + \delta c_N \leq z, \quad (63d)$$

where

$$\delta c_i = \sum_{p=1}^{lN} \delta c_i^p, \quad \forall i \in \mathbb{Z}_{[0, N]}, \quad (64)$$

and, for each $p \in \mathbb{Z}_{[1, lN]}$:

$$(u_i^p, x_i^p, \delta c_i^p) = 0, \quad \forall i \in \mathbb{Z}_{[0, k]} \quad (65a)$$

$$x_{k+1}^p = E_{(j)}, \quad (65b)$$

$$x_{i+1}^p - Ax_i^p - Bu_i^p = 0, \quad \forall i \in \mathbb{Z}_{[k+1, N-1]} \quad (65c)$$

$$\bar{C}x_i^p + \bar{D}u_i^p + H\delta c_i^p \leq 0, \quad \forall i \in \mathbb{Z}_{[k+1, N-1]} \quad (65d)$$

$$\bar{Y}x_N^p + H_f\delta c_N^p \leq 0. \quad (65e)$$

where $k = (p-j)/l$ and $j = 1 + (p-1) \bmod l$.

The decision variables in this problem are the nominal states and controls \hat{x}_i and v_i at each stage (the initial state \hat{x}_0 is known, hence *not* a decision variable), plus the perturbed states, controls, and constraint contractions terms x_i^p , u_i^p , and δc_i^p for each subproblem at each stage.

We can now state the following key result, proof of which follows directly from Proposition 2 and the discussion of this section.

Theorem 3. *The convex, tractable QP (62)–(65) is equivalent to the robust optimal control problems (16) and (29). The receding horizon control law $u = \mu_N(x)$ in (18) can be implemented using the solution to (62)–(65) as $u = v_0^*(x)$.*

It is important to note that the constraints in (63)–(65) can be rewritten in diagonalized form by interleaving the variables by time index. For the nominal problem, define the stacked vectors of variables:

$$x_0 := \text{vec}(v_0, \hat{x}_1, v_1, \dots, \hat{x}_{N-1}, v_{N-1}, \hat{x}_N). \quad (66)$$

For the p^{th} perturbation problem in (65), which corresponds to a unit disturbance at some stage k , define:

$$x_p := \text{vec}(u_{k+1}^p, \delta c_{k+1}^p, x_{k+2}^p, u_{k+2}^p, \delta c_{k+2}^p, \dots, x_{N-1}^p, u_{N-1}^p, \delta c_{N-1}^p, x_N^p, \delta c_N^p). \quad (67)$$

This yields a set of banded matrices A_0 , A_p , C_0 , and C_p , and right-hand-side vectors d_0 , d_p , b_0 , and b_p formed from the constraints in (63) and (65), and a set of coupling matrices J_p formed from the constraint coupling constraint (64). The result is a set of constraints in singly-bordered block-diagonal form with considerable structure and sparsity:

$$\begin{bmatrix} A_0 & & & & \\ & A_1 & & & \\ & & \ddots & & \\ & & & & A_{LN} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{LN} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{LN} \end{bmatrix}, \quad \begin{bmatrix} C_0 & J_1 & \cdots & J_{LN} \\ & C_1 & & \\ & & \ddots & \\ & & & C_{LN} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{LN} \end{bmatrix} \leq \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{LN} \end{bmatrix}. \quad (68)$$

For completeness the matrices A_0 , A_p , C_0 , and C_p , and vectors d_0 , d_p , b_0 , and b_p are defined in Appendix A. The coupling matrices J_p are easily constructed from the coupling equation (64). Note that only the vectors b_0 and d_0 are functions of x .

Remark 5. *It is possible to define a problem structure similar to that in (62)–(65) for the more general polytopic disturbance sets discussed in Section 4.1 via introduction of states in a similar manner. However, in this case the perturbation subproblems (65) contain an additional coupling constraint for the subproblems associated with each stage.*

6 Interior-Point Method for Robust Control

In this section we demonstrate, using a primal-dual interior-point solution technique, that the quadratic program defined in (62)–(65) is solvable in time cubic in the horizon length N at each iteration, when $n+m$ is dominated by N ; a situation common, for example, in the rapidly growing number of aerospace and automotive applications of predictive control [36, Sec. 3.3] [40]. This is a major improvement on the $O(N^6)$ work per iteration associated with the compact (dense) formulation (45), or the equivalent problem (50); cf. Remark 3. This computational improvement comes about due to the improved structure and sparsity of the problem. Indeed, akin to the situation in [41], we will show that each subproblem in the QP (62)–(65) has the same structure as that of an unconstrained optimal control problem without disturbances.

We first outline some of the general properties of interior-point solution methods.

6.1 General Interior-Point Methods

With a slight abuse of notation, we consider the general constrained quadratic optimization problem

$$\min_{\theta} \frac{1}{2} \theta^T Q \theta \quad \text{subject to } A\theta = b, \quad C\theta \leq d, \quad (69)$$

where the matrix Q is positive semidefinite. The Karush-Kuhn-Tucker conditions require that a solution θ^* to this system exists if and only if additional vectors π^* , λ^* and z^* exist that satisfy the following conditions:

$$Q\theta + A^T\pi + C^T\lambda = 0 \quad (70a)$$

$$A\theta - b = 0 \quad (70b)$$

$$-C\theta + d - z = 0 \quad (70c)$$

$$\lambda^T z = 0 \quad (70d)$$

$$(\lambda, z) \geq 0 \quad (70e)$$

The constraint $\lambda^T z = 0$ can be rewritten in a slightly more convenient form by defining diagonal matrices Λ and Z such that

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \quad (71)$$

so that $\Lambda Z \mathbf{1} = 0$. Primal-dual interior-point algorithms search for a solution to the KKT conditions (70) through repeated solutions of a set of Newton-like equations of the form

$$\begin{bmatrix} Q & A^T & C^T \\ A & & \\ C & & I \\ & \Lambda & Z \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\pi \\ \Delta\lambda \\ \Delta z \end{bmatrix} = - \begin{bmatrix} r_Q \\ r_A \\ r_C \\ r_Z \end{bmatrix}. \quad (72)$$

The particular choice of right hand sides for this equation is determined by the particular interior-point algorithm employed; the reader is referred to [51] for a thorough review. However, all such methods maintain the strict inequalities $(\lambda, z) > 0$ at each iteration. As a result, the matrices Λ and Z are guaranteed to remain full rank, and the system of equations in (72) can be simplified through elimination of the slack variables Δz , to form the reduced system

$$\begin{bmatrix} Q & A^T & C^T \\ A & & \\ C & & -\Lambda^{-1}Z \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\pi \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} r_Q \\ r_A \\ (r_C - \Lambda^{-1}r_Z) \end{bmatrix} \quad (73)$$

Since the number of interior-point iterations required in practise is only weakly related to the number of variables [51], the principal consideration is the time required to factor the Jacobian matrix (i.e., the matrix on the left-hand-side), and solve the linear system in (73).

6.2 Robust Control Formulation

For the robust optimal control problem described in (62)–(65), the system of equations in (73) can be arranged to yield a highly structured set of linear equations through appropriate ordering of the primal and dual variables and their Lagrange multipliers at each stage. As will be shown, this ordering enables the development of an efficient factorization procedure for the linear system in (73).

We use λ_i and λ_N to denote the Lagrange multipliers for the constraints (63c) and (63d) in the nominal system, and z_i and z_N for the corresponding slack variables. We similarly use λ_i^p and λ_N^p to denote the multipliers in (65d) and (65e) for the p^{th} perturbation subproblem, with slack variables z_i^p and z_N^p . We use π_i and π_i^p to denote the dual variables for (63) and (65).

The linear system (73) for the robust control problem (62)–(65) can then be reordered to form a symmetric, block-bordered, banded diagonal set of equations, by interleaving the primal and dual variables within the nominal and perturbed problems, while keeping the variables from each

where the matrix \mathfrak{B} is block-diagonal with banded blocks \mathcal{B}_p , and $\mathfrak{J} := [\mathcal{J}_1 \ \dots \ \mathcal{J}_{lN}]$. A block-partitioned matrix of this type can be factored and solved as

$$\begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\mathfrak{B}^{-1}\mathfrak{J}^T & I \end{bmatrix} \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & \mathfrak{B}^{-1} \end{bmatrix} \begin{bmatrix} I & -\mathfrak{J}\mathfrak{B}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{b}_A \\ \mathbf{b}_B \end{bmatrix}, \quad (81)$$

with

$$\Delta := (\mathcal{A} - \mathfrak{J}\mathfrak{B}^{-1}\mathfrak{J}^T). \quad (82)$$

where, by virtue of Lemmas 1 and 2, the matrix Δ is always full rank [25, Thm. 0.8.5].

The structure of the linear system in (77) is quite common [21, 31, 35], and can be solved using the following procedure based on Schur complements:

	Operation		Complexity	
<i>factor:</i>	$\mathcal{B}_i = L_i D_i L_i^T$	$\forall i \in \mathbb{Z}_{[1, lN]}$	$lN \cdot \mathcal{O}(\beta^3 N)$	(83a)
	$\Delta = \mathcal{A} - \left(\sum_{i=1}^{lN} \mathcal{J}_i \mathcal{B}_i^{-1} \mathcal{J}_i^T \right)$		$lN \cdot \mathcal{O}(\beta^3 N^2)$	(83b)
	$= L_\Delta D_\Delta L_\Delta^T$		$\mathcal{O}(\beta^3 N^3)$	(83c)
<i>solve:</i>	$\tilde{\mathbf{b}}_i = L_i^{-T} (D_i^{-1} (L_i^{-1} \mathbf{b}_i))$,	$\forall i \in \mathbb{Z}_{[1, lN]}$	$lN \cdot \mathcal{O}(\beta^2 N)$	(83d)
	$\mathbf{z}_A = \mathbf{b}_A - \sum_{i=1}^{lN} (\mathcal{J}_i \tilde{\mathbf{b}}_i)$,		$lN \cdot \mathcal{O}(\beta N)$	(83e)
	$\mathbf{x}_A = L_\Delta^{-T} (D_\Delta^{-1} (L_\Delta^{-1} \mathbf{z}))$,		$\mathcal{O}(\beta^2 N^2)$	(83f)
	$\mathbf{z}_i = \mathcal{J}_i^T \mathbf{x}_A$,	$\forall i \in \mathbb{Z}_{[1, lN]}$	$lN \cdot \mathcal{O}(\beta N)$	(83g)
	$\mathbf{x}_i = \tilde{\mathbf{b}}_i - L_i^{-T} (D_i^{-1} (L_i^{-1} \mathbf{z}_i))$.	$\forall i \in \mathbb{Z}_{[1, lN]}$	$lN \cdot \mathcal{O}(\beta^2 N)$	(83h)

□

Remark 6. *It is important to recognize that the order of operations in this solution procedure has a major influence on its efficiency. In particular, special care is required in forming the products $\mathcal{J}_i \mathcal{B}_i^{-1} \mathcal{J}_i^T$, particularly when the matrix \mathcal{J}_i^T is sparse, as many sparse factorization codes require that the right hand side vectors for a solve of the form $\mathcal{B}_i^{-1} \mathbf{b}$ be posed as dense columns. We note that, strictly speaking, the proposed method relies on the Riccati factorization technique discussed in Appendix B.2 for the factorization of the matrices \mathcal{B}_i , rather than factorization into $\mathcal{B}_i = L_i D_i L_i^T$, though this distinction is not material to our proof. For the formulation in (77) it is also important to note that since the coupling matrices \mathcal{J}_i have no more than a single 1 on every row and column, matrix products involving left or right multiplication by \mathcal{J}_i or \mathcal{J}_i^T do not require any floating point operations to calculate. The reader is referred to [8, App. C] for a more complete treatment of complexity analysis for matrix operations.*

Remark 7. *If the factorization procedure (83) is employed, then the robust optimization problem is an obvious candidate for parallel implementation.*

Remark 8. *It is not necessary to hand implement the suggested variable interleaving and block factorization procedure to realize the suggested block-bordered structure in (77) and $\mathcal{O}(N^3)$ behavior, as any reasonably efficient sparse factorization code may be expected to perform similar steps automatically; see [15]. Note that the ‘‘arrowhead’’ structure in (77) should be reversed (i.e. pointing down and to the right) in order for direct LDL^T factorization to produce sparse factors.*

7 Results

Two sparse QP solvers were used to evaluate the proposed formulation. The first, OOQP [19], uses a primal-dual interior-point approach configured with the sparse factorization code MA27

from the HSL library [26] and the OOQP version of the multiple-corrector interior-point method of Gondzio [20].

The second sparse solver used was the QP interface to the PATH [14] solver. This code solves mixed complementarity problems using an active-set method, and hence can be applied to the stationary conditions of any quadratic program. Note we are dealing with convex QPs, hence each optimization problem and its associated complementarity system have equivalent solution sets.

All results reported in this section were generated on a single processor machine, with a 3Ghz Pentium 4 processor and 1GB of RAM. We restrict our attention to sparse solvers as the amount of memory required in the size of the problems considered is prohibitively large for dense factorization methods.

A set of test cases was generated to compare the performance of the two sparse solvers using the (\mathbf{M}, \mathbf{v}) formulation in (45) with the decomposition based method of Section 5. Each test case is defined by its number of states n and horizon length N . The remaining problem parameters were chosen using the following rules:

- There are twice as many states as inputs.
- The constraint sets W , \mathcal{Z} and X_f represent randomly selected symmetric bounds on the states and inputs subjected to a random similarity transformation.
- The states space matrices A and B are randomly generated, with (A, B) controllable, and A stable.
- The dimension l of the generating disturbance is chosen as half the number of states, with randomly generated E of full column rank.
- All test cases have feasible solutions. The current state is selected such that at least some of the inequality constraints in (63c) are active at the optimal solution.

The average computational times required by each of the two solvers for the two problem formulations for a range of problem sizes are shown in Table 1. Each entry represents the average of ten test cases, unless otherwise noted.

It is clear from these results that, as expected, the decomposition-based formulation can be solved much more efficiently than the original (\mathbf{M}, \mathbf{v}) formulation for the robust optimal control problem in every case, and that the difference in solution times increases dramatically with increased problem size. Additionally, the decomposition formulation seems particularly well suited to the interior-point solver (OOQP), rather than the active set method (PATH). Nevertheless we expect the performance of active set methods to improve relative to interior-point methods when solving a sequence of similar QPs that would occur in predictive control, i.e., when a good estimate of the optimal active set is available at the start of computation. That is, interior-point methods are particularly effective in “cold start” situations, while the efficiency of active set methods is likely to improve given a “warm start”. As is common in interior-point methods, we find that the actual number of iterations required for solution of each problem type in Table 1 is nearly constant with increasing horizon length.

Figure 1 shows that the interior-point solution time increases cubically with horizon length for a randomly generated problem with 4 states. The performance closely matches the predicted behavior described in Section 5. For the particular problem shown, the number of iterations required for the OOQP algorithm to converge increased from 9 to 11 over the range of horizon lengths considered.

Table 1: Average Solution Times (sec)

Problem Size	(M, v)		Decomposition	
	OOQP	PATH	OOQP	PATH
2 states, 4 stages	0.005	0.003	0.005	0.001
2 states, 8 stages	0.023	0.019	0.019	0.016
2 states, 12 stages	0.064	0.060	0.039	0.195
2 states, 16 stages	0.191	0.206	0.079	0.456
2 states, 20 stages	0.444	0.431	0.141	0.702
4 states, 4 stages	0.026	0.047	0.021	0.033
4 states, 8 stages	0.201	0.213	0.089	0.117
4 states, 12 stages	0.977	2.199	0.287	2.926
4 states, 16 stages	3.871	39.83	0.128	10.93
4 states, 20 stages	12.99	76.46	1.128	31.03
8 states, 4 stages	0.886	4.869	0.181	1.130
8 states, 8 stages	7.844	49.15	0.842	19.59
8 states, 12 stages	49.20	303.7	2.949	131.6
8 states, 16 stages	210.5	x	7.219	x
8 states, 20 stages	501.7	x	13.14	x
12 states, 4 stages	4.866	24.66	0.428	6.007
12 states, 8 stages	95.84	697.1 [†]	3.458	230.5 [†]
12 states, 12 stages	672.2	x	11.86	x
12 states, 16 stages	x	x	33.04	x
12 states, 20 stages	x	x	79.06	x

x – Solver failed all test cases
[†] – Based on limited data set due to failures

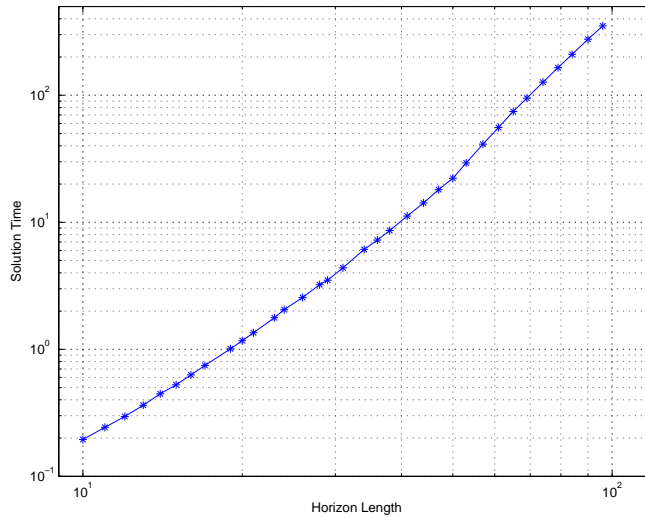


Figure 1: Computation time vs. horizon length for a 4 state system, using decomposition method and OOQP solver

8 Conclusions and Future Work

We have derived a highly efficient computational method for calculation of affine-state feedback policies for robust control of constrained systems with bounded disturbances. This is done by exploiting the structure of the underlying optimization problem and deriving an equivalent problem with considerable structure and sparsity, resulting in a problem formulation that is particularly suited to an interior-point solution method. As a result, robustly stabilizing receding horizon control laws based on optimal state-feedback policies have become practically realizable, even for systems of significant size or with long horizon lengths.

In Section 6 we proved that, when applying an interior-point solution technique to our robust optimal control problem, each iteration of the method can be solved using a number of operations proportional to the cube of the control horizon length. We appeal to the Riccati based factorization technique in [41] to support this claim. However, we stress that the results in Section 7, which demonstrate this cubic-time behavior numerically, are based on freely available optimization and linear algebra packages and *do not* rely on any special factorization methods.

A number of open research issues remain. It may be possible to further exploit the structure of our control problem by developing specialized factorization algorithms for the factorization of each interior-point step, e.g. through the parallel block factorization procedure alluded to in Remark 7. It may also be possible to achieve considerably better performance by placing further constraints on the structure of the disturbance feedback matrix \mathbf{M} , though this appears difficult to do if the attractive invariance and stability properties of the present formulation are to be preserved.

Many of the system-theoretic results developed in [22] hold for a fairly broad classes of disturbances and cost functions. For example, when the disturbance is Gaussian the problem may be modified to require that the state and input constraints hold with a certain pre-specified probability, and the probabilistic constraints converted to second-order cone constraints [8, pp. 157–8]. Alternatively, the cost function for the finite horizon control problem may require the minimization of the finite-horizon ℓ_2 gain of a system [28]. In all of these cases, there is a strong possibility that the underlying problem structure may be exploited to realise a substantial increase in computational efficiency.

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A Matrix Definitions

Let the matrices $\mathbf{A} \in \mathbb{R}^{n(N+1) \times n}$ and $\mathbf{E} \in \mathbb{R}^{n(N+1) \times nN}$ be defined as

$$\mathbf{A} := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{E} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ A & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I_n \end{bmatrix}. \quad (84)$$

We also define the matrices $\mathbf{B} \in \mathbb{R}^{n(N+1) \times mN}$, $\mathbf{C} \in \mathbb{R}^{(sN+r) \times n(N+1)}$ and $\mathbf{D} \in \mathbb{R}^{(sN+r) \times mN}$ as

$$\mathbf{B} := \mathbf{E}(I_N \otimes B), \quad \mathbf{C} := \begin{bmatrix} I_N \otimes C & 0 \\ 0 & Y \end{bmatrix}, \quad \mathbf{D} := \begin{bmatrix} I_N \otimes D \\ 0 \end{bmatrix}. \quad (85)$$

It is easy to check that the expression in (22) is equivalent to (44) with $F := \mathbf{CB} + \mathbf{D}$, $G := \mathbf{CE}$, $H := -\mathbf{CA}$, $c := [\mathbf{1}_N \otimes b]$. Writing the nominal constraint equation (63) in matrix form, the coefficient matrices \mathbf{A}_0 and \mathbf{C}_0 in (68) are:

$$\mathbf{A}_0 := \begin{bmatrix} B & -I \\ A & B & -I \\ & \ddots & \\ A & B & -I \end{bmatrix}, \quad \mathbf{C}_0 := \begin{bmatrix} D \\ C & D \\ & \ddots & \\ C & D \\ & & Y \end{bmatrix}, \quad (86)$$

with corresponding right hand sides

$$\mathbf{b}_0 := \text{vec}(-Ax, 0, 0, \dots, 0), \quad \mathbf{d}_0 := \text{vec}(b - Cx, b, \dots, b, z). \quad (87)$$

For the p^{th} perturbation problem in (65), which corresponds to a unit disturbance at some stage k , the coefficient matrices \mathbf{A}_p and \mathbf{C}_p in (65) become

$$\mathbf{A}_p := \begin{bmatrix} B & 0 & -I \\ A & B & 0 & -I \\ & \ddots & \\ A & B & 0 & -I & 0 \end{bmatrix}, \quad \mathbf{C}_p := \begin{bmatrix} \bar{D} & H \\ \bar{C} & \bar{D} & H \\ & \ddots & \\ \bar{C} & \bar{D} & H \\ & & \bar{Y} & H_f \end{bmatrix}, \quad (88)$$

with corresponding right hand sides

$$\mathbf{b}_p := \text{vec}(-AE_{(j)}, 0, \dots, 0), \quad \mathbf{d}_p := \text{vec}(0, 0, \dots, 0, 0). \quad (89)$$

B Rank of the Jacobian and Reduction to Riccati Form

B.1 Rank of the Robust Control Problem Jacobian (Proof of Lem. 1)

In this section we demonstrate that, for the robust optimal control problem as defined in (63)–(65), the interior point Jacobian matrix in (74) is always full rank. Recalling the discussion in Section 6.1, the Jacobian matrix is full rank if the only solution to the system

$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{A} & 0 & 0 \\ \mathbf{C} & 0 & -\Sigma^T \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta\pi \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (90)$$

satisfies $\Delta\theta = 0$, $\Delta\pi = 0$, and $\Delta\lambda = 0$, where $\Sigma > 0$, $\mathbf{Q} \geq 0$, and the coefficient matrices \mathbf{A} and \mathbf{C} come from the left hand sides of the robust control constraints defined in (68). The matrix \mathbf{Q} is easily constructed from Q and R in (15). It is important to recognize that the coefficient matrix in (74) is simply a reordering of the left-hand side of (90).

From the first two rows of this system,

$$\Delta\theta^T \mathbf{Q} \Delta\theta + (\Delta\theta^T \mathbf{A}^T) \Delta\pi + \Delta\theta^T \mathbf{C}^T \Delta\lambda = \Delta\theta^T \mathbf{Q} \Delta\theta + \Delta\theta^T \mathbf{C}^T \Delta\lambda = 0. \quad (91)$$

Incorporating the final block row, $\mathbf{C} \Delta\theta = \Sigma \Delta\lambda$, we have

$$\Delta\theta^T \mathbf{Q} \Delta\theta + \Delta\lambda^T \Sigma \Delta\lambda = 0. \quad (92)$$

Since $\Sigma > 0$ for a strictly interior point, we conclude that $\Delta\lambda = 0$. We also note that for a *general* QP, so long as the matrix \mathbf{Q} is at least positive semi-definite (possibly 0), then a *sufficient* condition for full rank of the coefficient matrix would be for \mathbf{C} to be full column rank and \mathbf{A} to be full row rank.

For our particular case, we note that since the term $\Delta\theta^T \mathbf{Q} \Delta\theta$ is strictly positive definite in the nominal control variables \mathbf{v} (since the control weight R is positive definite by assumption) it is easy to confirm, using the state update equation, that all of the nominal state and control variables arising from (63) are zero.

The last two block rows of the system then require $\mathbf{A} \Delta\theta = 0$ and $\mathbf{C} \Delta\theta = 0$, where the coefficient matrices \mathbf{A} and \mathbf{C} for the robust control problem originally come from the constraints in (65), i.e. they require $\mathbf{A}_p \mathbf{x}_p = 0$ and $\mathbf{C}_p \mathbf{x}_p = 0$ in (68), where the variables \mathbf{x}_p consist of the perturbed states x_i^p , controls u_i^p , and constraint contraction vectors δc_i^p at each stage. It is easy to verify that the constraint (65d) becomes

$$\bar{\mathbf{C}} x_i^p + \bar{\mathbf{D}} u_i^p + \bar{\mathbf{H}} \delta c_i^p = 0, \quad (93)$$

which, recalling the definitions of $\bar{\mathbf{C}}$, $\bar{\mathbf{D}}$, and $\bar{\mathbf{H}}$ in (60), implies $\delta c_k^p = 0$ at each stage. Additionally, if the perturbation problem corresponds to a disturbance at some stage k , then the constraint (65c) requires $\mathbf{D} u_{k+1}^p = 0$ if the matrix \mathbf{D} is full column rank³. Using the state update equation (65c), one can then conclude that $x_{k+1} = 0 \Rightarrow u_{k+1} = 0 \Rightarrow x_{k+2} = 0$, and so forth, such that each variable in each perturbed control problem is zero, and thus the vector $\Delta x = 0$.

All that remains from the first block row of the linear system is then $\mathbf{A}^T \Delta\pi = 0$. Since the matrix \mathbf{A} is full row rank (because $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{1N}$ are), we can conclude that $\Delta\pi = 0$, and the Jacobian is thus full rank at every interior point iteration.

B.2 Reduction to Riccati Form (Proof of Lem. 2)

We next demonstrate that the sub-matrices \mathcal{B}_p in (74) are also full rank. We do this by performing a single step of block elimination on the Lagrange multipliers λ_i^p , and demonstrate that the resulting matrix is banded and invertible via the Riccati recursion technique of [41] in $\mathcal{O}(\beta^3 N^3)$ operations.

³Note that the full rank column condition on \mathbf{D} is not strictly necessary — a less restrictive sufficient condition is $\text{null}(\mathbf{D}) \cap \text{null}(\mathbf{B}) = \{0\}$.

