Characterization of the solution to a constrained H_∞ optimal control problem*

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Abstract

This paper characterizes the solution to a finite horizon min-max optimal control problem where the system is linear and discrete-time with control and state constraints, and the cost quadratic; the disturbance is negatively costed, as in the standard H_∞ problem, and is constrained. The cost is minimized over control policies and maximized over disturbance sequences so that the solution yields a feedback control. It is shown that the value function is piecewise quadratic and the optimal control policy piecewise affine, being quadratic and affine, respectively, in polytopes that partition the domain of the value function.

Keywords: min-max, constrained, H_∞, parametric optimization, optimal control.

1 Introduction

Characterizations of solutions to constrained optimal control problems appeared in the papers [1–4] that deal with the constrained linear-quadratic problem, in the papers [4–7] and thesis [8] that deal with hybrid or piecewise

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affine systems, and in papers that deal with min-max optimal control problems [9–13]. In these papers it is shown that the value function is piecewise affine or piecewise quadratic (depending on the nature of the cost function in the optimal control problem) and the control law is piecewise affine, being quadratic or affine in polytopes that constitute a polytopic partition of the domain of the value function. When disturbances are present, it is necessary to compute the solution sequentially using dynamic programming as in [10]. In this paper, which is motivated by recent research on H_∞ model predictive control [14–21], we obtain an explicit characterization of the solution to a constrained, min-max optimal control problem and consider here the choice of terminal cost and constraint set to ensure stability of the closed loop system with receding horizon control. The term H_∞ is used somewhat loosely since we consider the min-max problem with fixed γ. We consider, therefore, the problem of controlling a linear, discrete-time system described by

$$x^+ = Ax + Bu + Gw, \qquad y = Cx + Du \tag{1.1}$$

where $x \in I\!R^n$ is the state, $u \in I\!R^m$ the control and $w \in I\!R^p$ an additive disturbance (the 'adversary'); x^+ is the successor state and $y \in I\!R^r$ is the costed output. We frequently write the system dynamics in (1.1) in the form

$$x^+ = f(x, u, w)$$

where $f(x, u, w) \triangleq Ax + Bu + Gw$. The system is subject to hard control and state constraints

$$u \in U, \qquad x \in X \tag{1.2}$$

where $U \subseteq I\!R^m$ is a (compact) polytope and $X \subseteq I\!R^n$ a polytope; each set contains the origin in its interior (the assumption that X is a polytope rather than a polyhedron[1] is made for simplicity). The disturbance w is constrained to lie in the polytope $W \subseteq I\!R^p$; W contains the origin in its interior.

Let $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ denote a control policy (sequence of control *laws*) over horizon N and let $\mathbf{w} \triangleq \{w_0, w_1, \ldots, w_{N-1}\}$ denote a sequence of disturbances. Also, let $\phi(i; x, \pi, \mathbf{w})$ denote the solution of (1.1) when the initial state is x at time 0, the control policy is π and the disturbance sequence is $\mathbf{w}$, so that $\phi(i; x, \pi, \mathbf{w})$ is the solution, at time i of

$$x_{i+1} = Ax_i + B\mu_i(x_i) + Gw_i \tag{1.3}$$
$$x_0 = x \tag{1.4}$$

[1] A polyhedron is a set described by a finite set of inequalities; a polytope is a bounded polyhedron.

The cost $V_N(x, \pi, \mathbf{w})$, if the initial state is x, the control policy π and the disturbance sequence $\mathbf{w}$, is

$$V_N(x, \pi, \mathbf{w}) \triangleq \sum_{i=0}^{N-1} \ell(x_i, u_i, w_i) + V_f(x_N) \tag{1.5}$$

where, for all i, $x_i \triangleq \phi(i; x, \pi, \mathbf{w})$ and $u_i \triangleq \mu_i(x_i)$; $V_f(\cdot)$ is a terminal cost that may be chosen, together with a terminal constraint set X_f defined below, to ensure stability of the resultant receding horizon controller (see §6). The stage cost $\ell(\cdot)$ is a quadratic function, positive definite in x and u, and negative definite in w:

$$\ell(x, u, w) \triangleq (1/2)|x|_Q^2 + (1/2)|u|_R^2 - (\gamma^2/2)|w|^2 \tag{1.6}$$

where $\gamma > 0$, $|z|_Z^2 \triangleq z'Zz$, and Q and R are positive definite. The stage cost may be expressed as

$$\ell(x, u, w) \triangleq (1/2)|y|^2 - (\gamma^2/2)|w|^2, \; y \triangleq Hz \tag{1.7}$$

where $z \triangleq (x, u)$ and H is a suitably chosen matrix ((x, u) should be interpreted as a column vector $(x', u')'$ in matrix expressions). The terminal cost $V_f(\cdot)$ is a quadratic function

$$V_f(x) \triangleq (1/2)|x|_{P_f}^2 \tag{1.8}$$

in which P_f is positive definite. The optimal control problem $\mathbb{P}_N(x)$ that we consider is

$$\mathbb{P}_N(x): \qquad V_N^0(x) = \inf_{\pi \in \Pi_N(x)} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, \pi, \mathbf{w}) \tag{1.9}$$

where $\mathcal{W} \triangleq W^N$, is the set of admissible disturbance sequences, and $\Pi_N(x)$ is the set of admissible policies, i.e. those policies that satisfy, for all $\mathbf{w} \in \mathcal{W} \triangleq W^N$, the state and control constraints (1.2), and the terminal constraint

$$x_N \in X_f. \tag{1.10}$$

Inclusion of the hard disturbance constraint $\mathbf{w} \in \mathcal{W}$ is necessary when state constraints are present since, otherwise, for any policy π chosen by the controller, we can expect that there exists a disturbance sequence $\mathbf{w}$ that transgresses the state constraint. The terminal constraint set is a polytope, containing the origin in its interior, that satisfies $X_f \subseteq X$, ensuring satisfaction of the state constraint at time N. Hence the set of admissible policies is

$$\Pi_N(x) \triangleq \{\pi \mid \phi(i; x, \pi, \mathbf{w}) \in X,\ \mu_i(\phi(i; x, \pi, \mathbf{w})) \in U, i = 0, 1, \ldots, N-1,\\ \phi(N; x, \pi, \mathbf{w}) \in X_f,\ \forall \mathbf{w} \in \mathcal{W}\} \quad (1.11)$$

Let X_N denote the set of initial states for which a solution to $\mathbb{P}_N(x)$ exists (the domain of $V_N^0(\cdot)$, the controllability set), i.e.

$$X_N \triangleq \{x \mid \Pi_N(x) \neq \emptyset\}. \quad (1.12)$$

In addition to characterizing the solution to a min-max optimal control problem that has not previously been characterized, this paper provides an improvement of the transformation procedure used in [4, 7, 8] to obtain a parametric solution to the optimal control problem; the improvement simplifies the determination of the polytopes in which the control law and value function are affine and quadratic respectively and avoids unnecessary subpartitioning of overlapping polytopes required in [4, 8].

2 Dynamic Programming for Constrained Problems

The solution to $\mathbb{P}_N(x)$ may be obtained as follows. For all $j \in \mathbb{N}_+ \triangleq \{1, 2, \ldots\}$, let problem $\mathbb{P}_j$, the partial return function $V_j^0(\cdot)$, and the controllability set X_j be defined as in (1.5)–(1.9) with j replacing N; j denotes "time-to-go". Then the sequences $\{V_j^0(\cdot), \kappa_j(\cdot), X_j\}$, where $\kappa_j(\cdot)$ denotes the optimal control law $\mu_{N-j}^0(\cdot)$ at time $i = N-j$, may be calculated recursively as follows [10, 22]:

$$V_j^0(x) = \min_{u \in U} \max_{w \in W} \{\ell(x, u, w) + V_{j-1}^0(f(x, u, w)) \mid f(x, u, W) \subseteq X_{j-1}\} \quad (2.1)$$

$$\kappa_j(x) = \arg \min_{u \in U} \max_{w \in W} \{\ell(x, u, w) + V_{j-1}^0(f(x, u, w)) \mid\\ f(x, u, W) \subseteq X_{j-1}\} \quad (2.2)$$

$$X_j = X \cap \{x \mid \exists u \in U \text{ such that } f(x, u, W) \subseteq X_{j-1}\} \quad (2.3)$$

with boundary conditions

$$V_0^0(x) = V_f(x), \qquad X_0 = X_f. \quad (2.4)$$

The condition $f(x, u, W) \subseteq X_{j-1}$ in (2.2) and (2.3) may be expressed as

$$Ax + Bu \in X_{j-1} \ominus GW \tag{2.5}$$

where $\ominus$ denotes Pontryagin set difference defined by $\mathcal{A} \ominus \mathcal{B} \triangleq \{x \mid \{x\} \oplus \mathcal{B} \subseteq \mathcal{A}\}$ ($\oplus$ denotes set addition). For each integer j let $Z_j \subseteq I\!R^n \times I\!R^m$ be defined by

$$Z_j \triangleq \{(x, u) \in X \times U \mid f(x, u, W) \subseteq X_j\} \tag{2.6}$$

so that, from (2.3),

$$X_j = \mathrm{Proj}_X Z_{j-1}.$$

Here, and in the sequel, if a set Z, say, lies in a product space $I\!R^n \times I\!R^m$, $\mathrm{Proj}_X : I\!R^n \times I\!R^m \to I\!R^n$ denotes the projection operator defined by $\mathrm{Proj}_X Z = \{x \mid \exists u \in I\!R^m \text{ such that } (x, u) \in Z\}$ ($I\!R^n$ is regarded as x-space). Similarly, if Φ is a set in the product space $I\!R^n \times I\!R^m \times I\!R^p$, $\mathrm{Proj}_Z \Phi$ denotes the set $\{z \mid \exists w \in I\!R^p \text{ such that } (z, w) \in \Phi\}$. We can now establish some preliminary properties of the solution to $\mathbb{P}_N$. To analyze $\mathbb{P}_N(x)$ it is convenient to introduce the functions $J_j^0(\cdot)$, $j = 1, 2, \ldots$, defined by

$$J_j^0(x, u) \triangleq \max_{w \in W} \{\ell(x, u, w) + V_j^0(f(x, u, w))\} \tag{2.7}$$

The recursive equations (2.1)-(2.3) may therefore be rewritten as

$$V_j^0(x) = \min_{u \in U} \{J_{j-1}^0(x, u) \mid f(x, u, W) \subseteq X_{j-1}\} \tag{2.8}$$

$$J_{j-1}^0(x, u) \triangleq \max_{w \in W} \{\ell(x, u, w) + V_{j-1}^0(f(x, u, w))\} \tag{2.9}$$

$$\kappa_j(x) = \arg\min_{u \in U} \{J_{j-1}^0(x, u) \mid f(x, u, W) \subseteq X_{j-1}\} \tag{2.10}$$

$$X_j = X \cap \{x \mid \exists u \in U \text{ st } f(x, u, W) \subseteq X_{j-1}\} \tag{2.11}$$

for $j = 1, \ldots, N$ with endpoint conditions $V_0(\cdot) = V_f(\cdot)$, $X_0 = X_f$. Under our assumptions the sets X_j and Z_j are compact. If $X_0 = X_f$ is robust control invariant, the sets X_j are nested ($X_j \supseteq X_{j-1}$ for all $j \geq 1$). For each j, the domain of $V_j^0(\cdot)$ includes X_j but we are only interested in its values on X_j; similarly, the domain of $J_j^0(\cdot)$ includes Z_j but we are only interested in its values on Z_j.

3 Parametric Optimization

We seek a parametric solution to problem $\mathbb{P}_N(x)$, i.e. a solution for all values of the parameter which, in this case, is the state x. More precisely, since we employ constrained dynamic programming, we seek a parametric solution to problems $\mathbb{P}_j(x)$ for all $j \in \{1, \dots, N\}$. First, we introduce a few useful definitions.

Definition 1 *For any positive integer J, $\mathcal{I}_J \triangleq \{1, 2, \dots, J\}$; for any set $\mathcal{X}$, $\mathcal{J}^{\mathcal{X}}$ denotes an index set associated with a partition of $\mathcal{X}$.*

Definition 2 *A set $\mathcal{P} = \{P_i \mid i \in \mathcal{J}\}$, for some index set $\mathcal{J}$, is called a polyhedral (polytopic) partition of a closed (compact) set $\mathcal{X}$ if $\mathcal{X} = \cup_{i \in \mathcal{J}} P_i$, and the sets $P_i,\ i \in \mathcal{J}$ are polyhedrons (polytopes) with non-empty interiors which are non-intersecting (*$\text{interior}(P_i) \cap \text{interior}(P_j) = \emptyset$ *for all $i, j \in \mathcal{J}, i \neq j$).*

Definition 3 *A function $V : \mathcal{X} \to I\!R$ is said to be continuous piecewise quadratic on a polyhedral (polytopic) partition $\mathcal{P} = \{P_i \mid i \in \mathcal{J}\}$ of $\mathcal{X}$ if it is* continuous *and satisfies*

$$V(x) = (1/2)|x|^2_{Q_i} + q_i' x + r_i, \qquad \forall x \in P_i,\ i \in \mathcal{J}$$

for some $Q_i, q_i, r_i,\ i \in \mathcal{J}$. Similarly, a function $\kappa : \mathcal{X} \to \mathcal{U}$ is said to be piecewise affine on a polyhedral partition $\mathcal{P} = \{P_i \mid i \in \mathcal{J}\}$ of $\mathcal{X}$ if it is continuous and satisfies

$$\kappa(x) = K_i x + k_i, \qquad \forall x \in P_i,\ i \in \mathcal{J},$$

for some $K_i, k_i,\ i \in \mathcal{J}$, where $\mathcal{P}$ has the properties specified above.

The dynamic programming recursion (2.8)-(2.11) requires the repeated solution of two prototype problems $\mathbb{P}_{\min}$ and $\mathbb{P}_{\max}$ defined next:

$$\mathbb{P}_{\min}(x): \qquad V^0(x) = \min_u \{J(x, u) \mid (x, u) \in \mathcal{Z}\} \tag{3.1}$$

$$\mathbb{P}_{\max}(z): \qquad J^0(z) = \max_w \{V(z, w) \mid w \in W\} \tag{3.2}$$

The minimizer in $\mathbb{P}_{\min}(x)$ and the maximizer in $\mathbb{P}_{\max}(z)$ are defined, respectively, by

$$\kappa(x) \triangleq \arg\min_u \{J(x, u) \mid (x, u) \in \mathcal{Z}\} \tag{3.3}$$

$$\nu(z) \triangleq \arg\max_w \{V(z, w) \mid w \in W\}. \tag{3.4}$$

Problem $\mathbb{P}_{\min}(x)$ is the prototype for Problem (2.8) with $V^0(x)$ replacing $V_j^0(x)$, $J(x, u)$ replacing $J_{j-1}^0(x, u)$, and $(x, u) \in \mathcal{Z}$ replacing the constraints $f(x, u, W) \subseteq X_{j-1}$ $(Ax + Bu \in X_{j-1} \ominus GW)$ *and* $u \in U$. Similarly Problem $\mathbb{P}_{\max}(z)$ is the prototype for Problem (2.9) with $J^0(z)$ replacing $J_{j-1}^0(z)$, $V(z, w)$ replacing $\ell(x, u, w) + V_{j-1}^0(f(x, u, w))$, and z replacing (x, u). We first obtain the parametric solution of $\mathbb{P}_{\min}$.

3.1 The minimization problem $\mathbb{P}_{\min}$

The solution to $\mathbb{P}_{\min}(x)$ has properties given in Proposition 1 that has a simpler hypothesis than previous versions of this result. For completeness, continuity of the control law is also proven.

Proposition 1 *Suppose $J : \mathcal{Z} \to I\!R$ is a strictly convex, continuous function and that $\mathcal{Z}$ is a polytope. Then, for all $x \in \mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$, the solution $\kappa(x)$ to $\mathbb{P}_{\min}(x)$ exists and is unique. The value function $V^0(\cdot)$ is strictly convex and continuous with domain $\mathcal{X}$, and the control law $\kappa(\cdot)$ is continuous on $\mathcal{X}$.*

Proof: For all $x \in \mathcal{X}$, $\mathcal{U}(x) \triangleq \{u \mid (x,u) \in \mathcal{Z}\}$ is convex and compact. Let $\Lambda := \{(\lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\}$. For all x_1, x_2 in $\mathcal{X}$, all $\lambda = (\lambda_1, \lambda_2) \in \Lambda$:

$$\begin{aligned} V^0(\lambda_1 x_1 + \lambda_2 x_2) &= \min_u \{J(\lambda_1 x_1 + \lambda_2 x_2, u) \mid (\lambda_1 x_1 + \lambda_2 x_2, u) \in \mathcal{Z}\} \\ &\leq J(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 u_1 + \lambda_2 u_2\},\ u_i \triangleq \kappa(x_i),\ i = 1, 2 \\ &= J(\lambda_1(x_1, u_1) + \lambda_2(x_2, u_2)) \end{aligned}$$

But $\lambda_1(x_1, u_1) + \lambda_2(x_2, u_2) \in \mathcal{Z}$ since $\mathcal{Z}$ is convex and $(x_i, u_i) \in \mathcal{Z}$, $i = 1, 2$. Since $J(\cdot)$ is strictly convex

$$\begin{aligned} V^0(\lambda_1 x_1 + \lambda_2 x_2) &\leq \lambda_1 J(x_1, u_1) + \lambda_2 J(x_2, u_2) \\ &= \lambda_1 V^0(x_1) + \lambda_2 V^0(x_2)\ \forall \lambda_1,\ \lambda_2 \in \Lambda \end{aligned}$$

where the last inequality is strict if $\lambda_1 \notin \{0, 1\}$ so that $V^0(\cdot)$ is strictly convex. Since $J(\cdot)$ is strictly convex, $\kappa(x)$ is unique at each $x \in \mathrm{Proj}_X \mathcal{Z}$.

The constraint $(x, u) \in \mathcal{Z}$ imposes an implicit state-dependent constraint $u \in \mathcal{U}(x)$ on u where the set-valued function $\mathcal{U}(\cdot)$ is defined by

$$\mathcal{U}(x) \triangleq \{u \mid (x, u) \in \mathcal{Z}\}.$$

We claim that $\mathcal{U}(\cdot)$ is continuous (both outer and inner semi-continuous on $\mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$, the domain of $\mathcal{U}(\cdot)$. By definition [23], the set-valued map $\mathcal{U}(\cdot)$ is outer semi-continuous at $x \in \mathcal{X}$ if $\mathcal{U}(x)$ is closed and if, for any compact set G such that $\mathcal{U}(x) \cap G = \emptyset$ there exists an $\varepsilon > 0$ such that $\mathcal{U}(x) \cap G = \emptyset$ for all $x' \in B(x, \varepsilon) \cap \mathcal{X}$. The set-valued map $\mathcal{U}(\cdot)$ is inner semi-continuous at $x \in \mathcal{X}$ if, for any open set $G \subseteq I\!R^m$ such that $G \cap \mathcal{U}(x) \neq \emptyset$, there exists an $\varepsilon > 0$ such that $G \cap \mathcal{U}(x') \neq \emptyset$ for all $x' \in B(x, \varepsilon) \cap \mathcal{X}$. Here $B^j(x, \varepsilon) \triangleq \{x' \in I\!R^j \mid |x' - x| \leq \varepsilon\}$. The set-valued map $\mathcal{U}(\cdot)$ is outer semi-continuous because its graph, $\mathcal{Z}$, is closed so that, given any sequence $\{(x_i, u_i)\}$ in $\mathcal{Z}$ ($u_i \in \mathcal{U}(x_i)$ for all i) such that $(x_i, u_i) \to (\bar{x}, \bar{u})$, we have $(\bar{x}, \bar{u}) \in \mathcal{Z}$ so that $\bar{u} \in \mathcal{U}(\bar{x})$. Hence $\mathcal{U}(\cdot)$ is outer semi-continuous [23]. We can establish inner semi-continuity using the following result [24] whose proof is given in the appendix.

Lemma 1 (Clarke). *Suppose $\mathcal{Z}$ is a polytope in $I\!R^n \times I\!R^m$ and let $\mathcal{X}$ denote its projection on $I\!R^n$ ($\mathcal{X} = \{x \mid \exists u \in I\!R^m$ such that $(x, u) \in \mathcal{Z}\}$). Let $\mathcal{U}(x) \triangleq \{u \mid (x, u) \in \mathcal{Z}\}$. Then there exists a $K > 0$ such that, for all $x, x' \in \mathcal{X}$, for all $u \in \mathcal{U}(x)$, there exists a $u' \in \mathcal{U}(x')$ such that $|u' - u| \leq K|x' - x|$.*

Let x, x' be arbitrary points in $\mathcal{X}$ and $\mathcal{U}(x)$ and $\mathcal{U}(x')$ the associated sets (Figure 1 illustrates the proof for two cases: $x = x_1$ and $x = x_2$). Let G be an open set such that $\mathcal{U}(x) \cap G \neq \emptyset$ and let u be an arbitrary point in $\mathcal{U}(x) \cap G$. Because G is open, there exist an $\varepsilon > 0$ such that $B(u, \varepsilon) \triangleq \{v \mid |v - u| \leq \varepsilon\} \subset G$. Let $\varepsilon' \triangleq \varepsilon/K$. From Lemma 1, there exists a $u' \in \mathcal{U}(x') \cap G$ for all $x' \in B(x, \varepsilon') \cap \mathcal{X}$. This implies $\mathcal{U}(x') \cap G \neq \emptyset$ for all $x' \in B(x, \varepsilon') \cap \mathcal{X}$, so that $\mathcal{U}(\cdot)$ is inner semi-continuous. ■

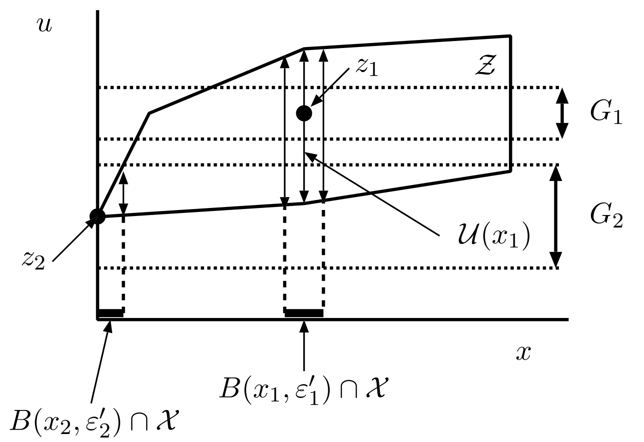


Figure 1: Inner semi-continuity of $\mathcal{U}(x)$

To solve the parametric problem $\mathbb{P}_{\min}$, we develop further the reverse transformation procedures proposed in [25] and utilized in [4, 7, 8]. We assume that $J(\cdot)$ is strictly convex and continuous piecewise quadratic on a polytopic partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ (for some index set $\mathcal{J}^{\mathcal{Z}}$) of $\mathcal{Z}$. For each $z = (x, u) \in \mathcal{Z}$, let $S^{\mathcal{Z}}(z)$, the index set of active polytopes at z, be defined by

$$S^{\mathcal{Z}}(z) \triangleq \{i \in \mathcal{J}^{\mathcal{Z}} \mid z \in P_i^{\mathcal{Z}}\} \tag{3.5}$$

so that $S^{\mathcal{Z}}(z)$ is the set of indices of active polytopes at z. Similarly, for each $x \in \mathcal{X} \triangleq \mathrm{Proj}_X(\mathcal{Z})$, let $S^0_{\mathcal{Z}}(x)$ be defined by

$$S^0_{\mathcal{Z}}(x) \triangleq S^{\mathcal{Z}}(x, \kappa(x)) \tag{3.6}$$

where $\kappa(x)$, the solution of $\mathbb{P}_{\min}(x)$, is defined by

$$\kappa(x) \triangleq \arg\min_u \{J(x, u) \mid (x, u) \in \mathcal{Z}\} \tag{3.7}$$

so that $S^0_{\mathcal{Z}}(x)$ is the index set of polytopes active at the solution $\kappa(x)$ of $\mathbb{P}_{\min}(x)$. For each $i \in \mathcal{J}^{\mathcal{Z}}$ we consider the simpler problem $\mathbb{P}_i(x)$ defined by

$$\mathbb{P}_i(x): \qquad V_i^0(x) = \min_u \{J_i(x, u) \mid (x, u) \in P_i^{\mathcal{Z}}\} \tag{3.8}$$

$$\kappa_i(x) = \arg\min_u \{J_i(x, u) \mid (x, u) \in P_i^{\mathcal{Z}}\} \tag{3.9}$$

where $J(z) = J_i(z)$ on $P_i^{\mathcal{Z}}$ and

$$J_i(z) = (1/2)z'Q_i z + q_i' z + s_i \tag{3.10}$$

for some Q_i, q_i, s_i, all $z = (x, u) \in P_i^{\mathcal{Z}}$. For each i, problem $\mathbb{P}_i(x)$ is a quadratic program since $J(\cdot)$ is quadratic on the polytope $P_i^{\mathcal{Z}}$.

Proposition 2 *Suppose $J : \mathcal{Z} \to I\!R$ is strictly convex and continuous and that $\mathcal{Z}$ is a polytope with a polytopic partition $\{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$. Then, for all $x \in P_i^{\mathcal{X}} \triangleq \mathrm{Proj}_X P_i^{\mathcal{Z}}$, all $i \in \mathcal{J}^{\mathcal{Z}}$, the solution $\kappa_i(x)$ to $\mathbb{P}_i(x)$ exists and is unique. Moreover, the value function $V_i^0(\cdot)$ is strictly convex and continuous, and $\kappa_i(\cdot)$ is continuous, in $P_i^{\mathcal{X}}$.*

The proof of Proposition 2 is almost identical to the proof of Proposition 1 noting that, for each $i \in \mathcal{J}^{\mathcal{Z}}$, $P_i^{\mathcal{Z}}$ is a polytope. Problem $\mathbb{P}_i(x)$, although simpler, is an artificial problem since the constraint $(x, u) \in P_i^{\mathcal{Z}}$ does not appear in the original problem $\mathbb{P}_{\min}(x)$, so that it is not obvious how best to relate the solutions to problems $\mathbb{P}_i(x)$, $i \in \mathcal{J}^{\mathcal{Z}}$, to the solution of $\mathbb{P}_{\min}(x)$. This difficulty was not totally satisfactorily dealt with in the literature quoted above but is resolved in the following result.

Proposition 3 *Suppose $J : \mathcal{Z} \to I\!R$ is continuous piecewise quadratic and strictly convex on a polytopic partition $\mathcal{P}^{\mathcal{Z}}$ of $\mathcal{Z}$. Then u is optimal for the minimization problem $\mathbb{P}_{\min}(x)$ if and only if u is optimal for the problems $\mathbb{P}_i(x)$ (i.e. if and only if $\kappa(x) = \kappa_i(x)$) for all $i \in S^0_{\mathcal{Z}}(x)$.*

Proof: Suppose $u = \kappa(x)$ is optimal for $\mathbb{P}_{\min}(x)$ but that, contrary to what is to be proven, there exists an $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$ such that u is not optimal for $\mathbb{P}_i(x)$. Let $u_i = \kappa_i(x)$ denote the solution of $\mathbb{P}_i(x)$. By definition, $(x, u_i) \in P_i^{\mathcal{Z}}$ and $(x, u) \in P_i^{\mathcal{Z}}$ (since $(x, u) \in P_j^{\mathcal{Z}}$ for all $j \in \mathcal{S}^0_{\mathcal{Z}}(x)$). Hence $u_i = \kappa_i(x)$ satisfies $V_i^0(x) = J_i(x, u_i) = J(x, u_i) < J_i(x, u) = J(x, u) = V^0(x)$ where we have made use of the fact that $J(x, v) = J_i(x, v)$ if $(x, v) \in P_i^{\mathcal{Z}}$. Hence $J(x, u_i) < V^0(x)$ which contradicts the optimality of u for $\mathbb{P}_{\min}(x)$. Suppose, next, that u is optimal for $\mathbb{P}_i(x)$ for all $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$ (so that $\kappa_i(x) = u$ for all $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$) but that, contrary to what is to be proved, u is not optimal for $\mathbb{P}_{\min}(x)$ so that there there exists a u^* satisfying $(x, u^*) \in \mathcal{Z}$ and $J(x, u^*) < J(x, u)$. Because $u \in P_i^{\mathcal{Z}}$ for all $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$ and $d(u, P_j^{\mathcal{Z}}) > 0$ for all $j \in \mathcal{J}^{\mathcal{Z}} \setminus \mathcal{S}^0_{\mathcal{Z}}(x)$, there exists a $\lambda \in (0, 1]$ and an $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$ such that $u_\lambda \triangleq u+\lambda(u^*-u)$ satisfies $(x, u_\lambda) \in P_i^{\mathcal{Z}}$. Since $u \mapsto J(x, u)$ is convex and $J(x, u^*) < J(x, u)$ it follows that $J(x, u_\lambda) < J(x, u)$. But $J(x, u_\lambda) = J_i(x, u_\lambda)$ (since $(x, u_\lambda) \in P_i^{\mathcal{Z}}$) and $J(x, u) = J_i(x, u)$ (since $(x, u) \in P_i^{\mathcal{Z}}$) so that $J_i(x, u_\lambda) < J_i(x, u)$, a contradiction of the optimality of u for $\mathbb{P}_i(x)$ for all $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$. ■

Proposition 3 shows that the solution to $\mathbb{P}_{\min}(x)$ is also the solution to a set of quadratic programs, namely $\mathbb{P}_i(x)$ for $i \in \mathcal{S}^0_{\mathcal{Z}}(x)$. We now analyse problem $\mathbb{P}_i(x)$ in more detail. Suppose that, for each $i \in \mathcal{J}^{\mathcal{Z}}$, polytope $P_i^{\mathcal{Z}}$ is defined by

$$P_i^{\mathcal{Z}} \triangleq \{z = (x, u) \mid M_i u \leq N_i x + p_i\} \tag{3.11}$$

where M_i, N_i, p_i each have r_i rows, so that

$$\begin{aligned} \mathbb{P}_i(x): \qquad & V_i^0(x) = \min_u \{J_i(x, u) \mid M_i u \leq N_i x + p_i\} \\ & \kappa_i(x) = \arg\min_u \{J_i(x, u) \mid M_i u \leq N_i x + p_i\}. \end{aligned} \tag{3.12}$$

The j^{th} constraint $M_i^j u \leq N_i^j x + p_i^j$ is said to be active at (x, u) if $M_i^j u = N_i^j x + p_i^j$. The set of active constraints for $\mathbb{P}_i(x)$ is $I_i^0(x)$, the set of constraints active at $(x, \kappa_i(x))$, so that

$$I_i^0(x) \triangleq \{j \in \mathcal{I}_{r_i} \mid M_i^j \kappa_i(x) = N_i^j x + p_i^j\}. \tag{3.13}$$

where the superscript j on a matrix (or vector) denotes the j^{th} row of the matrix (or vector). It follows from the definition of $\kappa_{i,I}(x) = K_{i,I} x + k_{i,I}$ that $I_i^0(x) = I$ for all $x \in \text{interior}\{X\}_{i,I}$ and that $I_i^0(x) \subseteq I$ on the boundary of $X_{i,I}$. The solution to $\mathbb{P}_i(x)$ is simple if the set of active constraints $I_i^0(x)$ for the problem is known in advance [3,26]. Suppose therefore the set of active constraints for $\mathbb{P}_i(x)$ at $(x, \kappa_i(x))$ ($\kappa_i(x)$ is the solution of $\mathbb{P}_i(x)$) is known, apriori, to be I, i.e. $I^0(x) = I$. Then $\mathbb{P}_i(x)$ is replaced by the simpler, equality constrained, problem

$$\mathbb{P}_{i,I}(x): \qquad V^0_{i,I}(x) = \min_u \{J_i(x,u) \mid M_i^j u = N_i^j x + p_i^j,\ j \in I\} \tag{3.14}$$

This is a quadratic optimization problem with affine equality constraints; the solution to this problem has, as is well known, the form

$$V^0_{i,I}(x) = (1/2)x' Q_{i,I} x + q'_{i,I} x + s_{i,I} \tag{3.15}$$

$$\kappa_{i,I}(x) = K_{i,I} x + k_{i,I} \tag{3.16}$$

for some $Q_{i,I}$, $s_{i,I}$, $K_{i,I}$ and $k_{i,I}$. Let M_i^I denote the matrix with rows M_i^j, $j \in I$. Let $PC_{i,I} \triangleq \{(M_i^I)'\lambda \mid \lambda \geq 0\} \subseteq I\!R^m$ denote the polar cone at 0 to the cone $\mathcal{F}_{i,I} = \{h \mid M_i^j h \leq 0,\ j \in I\}$ of feasible directions h for problem $\mathcal{P}_i(x)$ at $u = \kappa_{i,I}(x)$. The polar cone depends *solely* on I, the set of active constraints; it does not depend on the parameter x; also [4] $-\nabla_u J_i(x, \kappa_{i,I}(x)) \in PC_{i,I}$ if and only if $\langle \nabla_u J_i(x, \kappa_{i,I}(x)), h\rangle \geq 0$ for all feasible directions h ($h \in \mathcal{F}_{i,I}$), i.e. if and only if $\kappa_{i,I}(x)$ is optimal for the problem $\min_u \{J_i(x,u) \mid M_i^j u = N_i^j x + p_i^j,\ j \in I, M_i^j u \leq N_i^j x + p_i^j,\ j \in \mathcal{I}_{r_i} \setminus I\}$. For each $i \in \mathcal{J}^{\mathcal{Z}}$, let $P_i^{\mathcal{X}}$ denote the polytope defined by

$$P_i^{\mathcal{X}} \triangleq \{x \mid \exists u \text{ s.t. } (x,u) \in P_i^{\mathcal{Z}}\} = \mathrm{Proj}_X(P_i^{\mathcal{Z}}) \tag{3.17}$$

The polytope $P_i^{\mathcal{X}}$ is the domain of $V_i^0(\cdot)$. The following result holds [4]:

Proposition 4 *The affine control law $\kappa_{i,I}(\cdot)$ is optimal for problem $\mathbb{P}_i(x)$, at all x in the polytope $X_{i,I} \subseteq P_i^{\mathcal{X}}$ defined by*

$$X_{i,I} \triangleq \left\{ x \in X \;\middle|\; \begin{array}{rcl} M_i^j(K_{i,I}x + k_{i,I}) & \leq & N_i^j x + p_i^j,\ j \in \mathcal{I}_{r_i} \setminus I \\ -\nabla_u J_i(x, \kappa_{i,I}(x)) & \in & PC_{i,I} \end{array} \right\} \tag{3.18}$$

The restriction $x \in X$ is included in the definition of $X_{i,I}$ since it is not included as a constraint in $\mathbb{P}_i(x)$). Since the affine control law $\kappa_{i,I}(\cdot)$ is such that the equality constraint $M_i \kappa_{i,I}(x) = N_i x + p_i$ is satisfied for all x, and since the last inequality in (3.18) ensures that $\kappa_{i,I}(x)$ is optimal for $\min_u \{J_i(x,u) \mid M_i^j u = N_i^j x + p_i^j,\ j \in I, M_i^j u \leq N_i^j x + p_i^j,\ j \in \mathcal{I}_{r_i} \setminus I\}$, it follows that $\kappa_{i,I}(x)$ is optimal for $\mathbb{P}_i(x)$ in the polytope $X_{i,I}$. Thus [3,4,26] the solution to $\mathbb{P}_i(x)$ is affine, and the value function quadratic, in each polytope $X_{i,I}$; the set of all such non-empty polytopes (as I ranges over the subsets of $\{1, 2, \ldots, r_i\}$ constitute a polytopic partition of $\mathrm{Proj}_X(P_i^{\mathcal{Z}})$

so that the solution $\kappa_i(\cdot)$ to $\mathbb{P}_i(x)$ is piecewise affine, and the value function $V_i^0(\cdot)$ is piecewise quadratic, on this polytopic partition.

However, in our case, since we have to 'marry' a set of polytopes X_{i,I_i} for all i such that polytope $P_i^{\mathcal{Z}}$ is active (i.e. $i \in \mathcal{S}_{\mathcal{Z}}^0(x)$), is active, it is preferable to parameterize the polytopes in which the solution to $\mathbb{P}_i(x)$ is affine by the state $\bar{x}$, say, at which I_i is active rather than by the set I_i of active constraints. Also, for each $\bar{x} \in \mathcal{X} = \mathrm{Proj}_X(\mathcal{Z})$, let the polytope $X_i(\bar{x}) \subseteq P_i^{\mathcal{X}}$ be defined by

$$X_i(\bar{x}) \triangleq X_{i,I_i^0(\bar{x})} \tag{3.19}$$

where, for each set $I \subseteq \mathcal{J}^{\mathcal{Z}}$, $X_{i,I}$ is defined by (3.18). It follows from (3.18), with I replaced by $I_i^0(\bar{x})$, that $\bar{x} \in X_i(\bar{x})$. It was shown above that $I_i^0(x) = I_i^0(\bar{x})$ for all x in the interior of $X_i(\bar{x})$; it follows from (3.19) that $X_i(x) = X_i(\bar{x})$ for all x in the interior of $X_i(\bar{x})$.

The polytope $X(\bar{x})$ that figures in the parametric solution of $\mathbb{P}_{\min}(x)$ is defined, for each $\bar{x} \in \mathcal{X}$, by

$$X(\bar{x}) \triangleq \bigcap \{X_i(\bar{x}) \mid i \in S_{\mathcal{Z}}^0(\bar{x})\} \tag{3.20}$$

Clearly $\bar{x} \in X(\bar{x})$ and $X(x') = X(\bar{x})$ for all $x' \in X(\bar{x})$. For each $\bar{x} \in \mathcal{X}$, let the functions $V_{\bar{x}}^0(\cdot)$ and $\kappa_{\bar{x}}(\cdot)$ be defined on $X(\bar{x})$ by

$$V_{\bar{x}}^0(x) \triangleq V_i^0(x),\ \forall i \in S_{\mathcal{Z}}^0(\bar{x}) \tag{3.21}$$

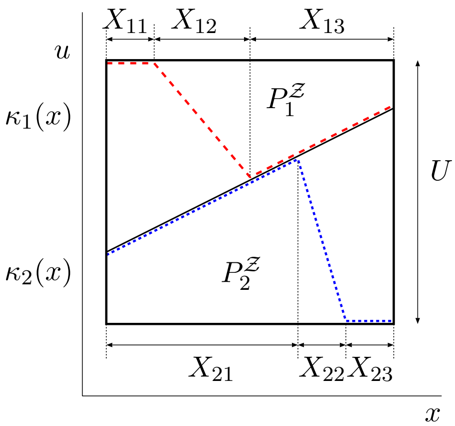
$$\kappa_{\bar{x}}(x) \triangleq \kappa_i(x),\ \forall i \in S_{\mathcal{Z}}^0(\bar{x}) \tag{3.22}$$

The domain of each function is $X(\bar{x})$; that the functions are well defined follows from Proposition 3 and equations (3.19) and (3.20) which show that $\kappa_i(x) = \kappa_j(x)$ for all $i, j \in S_{\mathcal{Z}}^0(\bar{x})$, all $x \in X(\bar{x})$. Summarizing, we have:

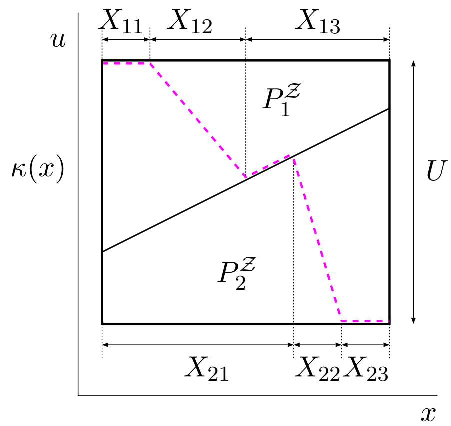
Theorem 1 *Suppose $J : \mathcal{Z} \to I\!R$ is continuous piecewise quadratic and strictly convex on a polytopic partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of $\mathcal{Z}$ and that, for each $i \in \mathcal{J}^{\mathcal{Z}}$, $P_i^{\mathcal{Z}}$ has a non-empty interior. Then the value function $V^0(\cdot)$ is continuous piecewise quadratic and strictly convex on a polytopic partition $\mathcal{P}^{\mathcal{X}} = \{P_i^{\mathcal{X}} \mid i \in \mathcal{J}^{\mathcal{X}}\}$ of $\mathcal{X}=$ $\mathrm{Proj}_X(\mathcal{Z})$. The minimizer $\kappa(\cdot)$ is piecewise affine on $\mathcal{P}^{\mathcal{X}}$. The polytopes $P_i^{\mathcal{X}}$ are each of the form $X(\bar{x})$ for some $\bar{x} \in \mathcal{X}$; the value function and optimal control law satisfy $V^0(x) = V_{\bar{x}}^0(x)$ and $\kappa(x) = \kappa_{\bar{x}}(x)$ for all $x \in X(\bar{x})$ and some $\bar{x} \in \mathcal{X}$.*

The proof of this result follows from Propositions 3 and 4 and the discussion above. The result is illustrated in Figure 2 for the simple case when $\mathcal{Z} =$

$\mathcal{P}_1^{\mathcal{Z}} \cup \mathcal{P}_2^{\mathcal{Z}}$ has two partitions $\mathcal{P}_1^{\mathcal{Z}}$ and $\mathcal{P}_2^{\mathcal{Z}}$ in each of which $J(\cdot)$ is quadratic. Problem $\mathcal{P}_1^{\mathcal{Z}}(x)$ is, therefore, a parametric quadratic program; its solution $\kappa_1(x)$ is known to be piecewise affine on a polytopic partition of $\mathcal{P}_1^{\mathcal{X}}$; in Figure 2, $\mathcal{P}_1^{\mathcal{X}} = \mathcal{X}$ and the polytopic partition is $\{X_{11}, X_{12}, X_{13}\}$. The solution to $\mathcal{P}_1^{\mathcal{Z}}(x)$ is $\kappa_1(x)$ which is affine in each of the polytopes X_{11}, X_{12} and X_{13}. Similarly the solution to the quadratic program $\mathcal{P}_1^{\mathcal{Z}}(x)$ is $\kappa_2(x)$ that is piecewise affine on a polytopic partition $\{X_{21}, X_{22}, X_{23}\}$ of $\mathcal{P}_2^{\mathcal{X}} = \mathcal{X}$. The sets X_{ij} and the optimal control laws $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are shown in the Figure. At each $x \in \mathcal{X}$, there are two candidates $\kappa_1(x)$ and $\kappa_2(x)$ for the optimal control $\kappa(x)$ for the original problem $\mathbb{P}_{\min}(x)$. Theorem 1 resolves this difficulty; at all z on the boundary between $P_1^{\mathcal{Z}}$ and $P_2^{\mathcal{Z}}$, $\mathcal{S}^{\mathcal{Z}}(z) = \{1, 2\}$ since both polytopes are active. Hence on the boundary, a control u is optimal for the original problem $\mathbb{P}_{\min}(x)$ if and only if it is optimal for both problems $\mathcal{P}_1^{\mathcal{Z}}(x)$ and $\mathcal{P}_2^{\mathcal{Z}}(x)$. At all $x \in X_{11} \cup X_{12}$ (except at its intersection with X_{13}), only $(x, \kappa_2(x))$ lies on the boundary between $P_1^{\mathcal{Z}}$ and $P_2^{\mathcal{Z}}$ ($(x, \kappa_1(x))$ does not lie on this boundary). Thus, in $X_{11} \cup X_{12}$, the optimal control is not $\kappa_2(x)$); it is $\kappa(x) = \kappa_1(x)$. At all $x \in X_{13} \cap X_{21}$ both $(x, \kappa_1(x))$ and $(x, \kappa_2(x))$ lie on the boundary between $P_1^{\mathcal{Z}}$ and $P_2^{\mathcal{Z}}$ so that the optimal control here is $\kappa(x) = \kappa_1(x) = \kappa_2(x)$. Finally, at all $x \in X_{22} \cup X_{23}$ (except at its intersection with X_{21}), only $(x, \kappa_1(x))$ lies on the boundary between $P_1^{\mathcal{Z}}$ and $P_2^{\mathcal{Z}}$ ($(x, \kappa_2(x))$ does not lie on this boundary); thus, in $X_{22} \cup X_{23}$, the optimal control is $\kappa(x) = \kappa_2(x)$. Hence $\kappa(\cdot)$ is completely defined on $\mathcal{X}$. This procedure avoids the overlapping of sets that results in previous analysis of this problem.



(a) Solutions of $\mathbb{P}_i(x)$, $i = 1, 2$



(b) Solution of $\mathbb{P}_{\min}(x)$

Figure 2: Problem $\mathbb{P}(x)$

3.2 The maximization subproblem $\mathbb{P}_{\max}$

In the minimization subproblem (3.1), the function $V(\cdot)$ being minimized is convex in both x and u. In contrast, in the maximization subproblem (3.2), the function $V(\cdot)$ being maximized is convex in z and (under suitable conditions) concave in w. Hence we proceed somewhat differently.

Proposition 5 *Suppose $V : \Phi \to I\!R$ is such that $z \mapsto V(z, w)$ is strictly convex and continuous for each $w \in W$, $w \mapsto V(z, w)$ is strictly concave and continuous for each z in $\mathcal{Z} \triangleq \mathrm{Proj}_Z(\Phi)$, and that Φ is a* polytope *with a non-empty interior. Then, for all $(z, w) \in \Phi$, the solution $\nu(z)$ to $\mathbb{P}_{\max}(z)$ exists and is unique. Moreover, the value function $J^0(\cdot)$ is strictly convex and continuous with domain $\mathcal{Z}$, and $\nu(\cdot)$ is continuous in $\mathcal{Z}$.*

Proof: Since $J^0(\cdot)$ is the maximum of a set of strictly convex and continuous functions, it is also strictly convex and continuous. The existence and uniqueness of $\nu(z)$ for each $z \in \mathcal{Z}$ follows from the strict concavity and continuity of $w \mapsto V(z, w)$ and the compactness of W. The continuity of $\nu(\cdot)$ follows from the uniqueness of $\nu(z)$ at each z (Theorem 5.4.3 in [23]). ■

To obtain a parametric solution to $\mathbb{P}_{\max}$, we assume that $V(\cdot)$ is continuous piecewise quadratic on a polytopic partition $\mathcal{P}^\Phi = \{P_i^\Phi \mid i \in \mathcal{J}^\Phi\}$ of $\Phi \triangleq \mathcal{Z} \times W$ (in the absence of additional restrictions, $\mathcal{Z} = X \times U$ so both $\mathcal{Z}$ and Φ are polytopic). For each $(z, w) \in \Phi$, let $S^\Phi(z, w)$, the index set of active polytopes at (z, w), be defined by

$$S^\Phi(z, w) \triangleq \{i \in \mathcal{J}^\Phi \mid (z, w) \in P_i^\Phi\} \tag{3.23}$$

so that $S^\Phi(z, w)$ is the set of indices of active polytopes at (z, w). Similarly, for each $z \in \mathcal{Z} \triangleq \mathrm{Proj}_Z(\Phi)$, let $S_\Phi^0(z)$ be defined by

$$S_\Phi^0(z) \triangleq S^\Phi(z, \nu(z)) \tag{3.24}$$

where

$$\nu(z) \triangleq \arg\max_w \{V(z, w) \mid w \in W\} \tag{3.25}$$

so that $S_\Phi^0(z)$ is the index set of polytopes active at the solution $\nu(z)$ of $\mathbb{P}_{\max}(z)$. For each $i \in \mathcal{J}^\Phi$, each $z \in \mathcal{Z}$, we define the simpler problem $\mathbb{P}_i(z)$ defined by

$$\mathbb{P}_i(z): \qquad J_i^0(z) = \max_w \{V_i(z, w) \mid (z, w) \in P_i^\Phi\} \tag{3.26}$$

$$\nu_i(z) = \arg\max_w \{V_i(z, w) \mid (z, w) \in P_i^\Phi\} \tag{3.27}$$

where $V(z,w) = V_i(z,w)$ on P_i^Φ and $V_i(\cdot)$ is quadratic. For each i, problem $\mathbb{P}_i^\Phi(z)$ is a quadratic program.

Proposition 6 *Suppose $V : \Phi \to I\!R$ is strictly concave (hence continuous) in w for each $z \in \mathrm{Proj}_Z\Phi$, and that Φ is a polytope with a polytopic partition $\{P_i^\Phi \mid i \in \mathcal{J}^\Phi\}$ such that, for each $i \in \mathcal{J}^\Phi$, P_i^Φ has a non-empty interior. Then, for all $z \in P_i^{\mathcal{Z}} = \mathrm{Proj}_Z P_i^\Phi$, all $i \in \mathcal{J}^\Phi$, the solution $\nu_i(z)$ to $\mathbb{P}_i(z)$ exists and is unique. Moreover, the value function $J_i^0(\cdot)$ is strictly convex (hence continuous) with domain $P_i^{\mathcal{Z}}$, and $\nu_i(\cdot)$ is continuous at any $z \in P_i^{\mathcal{Z}}$.*

The proof of Proposition 6 is similar to the proof of Proposition 5. The relation between the solution to $\mathbb{P}_{\max}$ and the solutions to the subproblems $\mathbb{P}_i^\Phi$, $i \in \mathcal{J}^\Phi$ is given in the next result.

Proposition 7 *Suppose $V : \Phi \to I\!R$ is continuous piecewise quadratic, strictly convex in z and strictly concave in w and is continuous piecewise quadratic in a polytopic partition $\mathcal{P}^\Phi = \{P_i^\Phi \mid i \in \mathcal{J}^\Phi\}$ of the polytope Φ. Then w is optimal for the maximization problem $\mathbb{P}_{\max}(z)$ if and only if w is optimal for the problems $\mathbb{P}_i(z)$ ($\nu(z) = \nu_i(z)$) for all $i \in S_\Phi^0(z)$.*

The proof of Proposition 7 is similar to the proof of Proposition 3. We now exploit the continuous piecewise quadratic nature of $V(\cdot)$. For each $i \in \mathcal{J}^\Phi$, subproblem $\mathbb{P}_i(z)$ may be expressed as:

$$\mathbb{P}_i(z): \qquad J_i^0(z) = \max_w \{V_i(z,w) \mid M_i w \leq N_i z + p_i\} \tag{3.28}$$

($V_i(\cdot)$ is quadratic) for some M_i, N_i, p_i, each matrix (vector) having r_i rows. If we assume that the constraints indexed by $I \subseteq \mathcal{I}_{r_i}$ are active, then $\mathbb{P}_i(z)$ is replaced by the simpler, equality constrained, problem

$$\mathbb{P}_{i,I}(z): \qquad J_i^0(z) = \max_w \{V_i(z,w) \mid M_i^j w = N_i^j z + p_i^j,\ j \in I\} \tag{3.29}$$

where the superscript j on matrix (or vector) denotes the j^{th} row of the matrix (or vector). The solution to this problem is

$$J_{i,I}^0(z) = (1/2) z' Q_{i,I} z + q_{i,I}' z + s_{i,I} \tag{3.30}$$

$$\nu_{i,I}(z) = K_{i,I} z + k_{i,I} \tag{3.31}$$

Let M_i^I denote the matrix the rows of which are M_i^j, $j \in I$. Let $PC_{i,I} \triangleq \{(M_i^I)'\lambda \mid \lambda \geq 0\} \subseteq I\!R^p$ denote the polar cone at 0 to the cone $\mathcal{F}_{i,I} = \{h \mid M_i^j h \leq 0,\ j \in I\}$ of feasible directions h for problem $\mathbb{P}_i(z)$ at $w = \nu_{i,I}(z)$; the polar cone depends *solely* on I, the set of active constraints; it does not depend on the parameter z. The following result holds [4]:

Proposition 8 *The affine control law $\nu_{i,I}(\cdot)$ is optimal for problem $\mathcal{P}_i(z)$ at all z in the polytope $Z_{i,I}$ defined by*

$$Z_{i,I} \triangleq \left\{ z \in \mathcal{Z} \;\middle|\; \begin{array}{rcl} M_i^j(K_{i,I}z + k_{i,I}) & \leq & N_i^j z + p_i^j, \; j \in \mathcal{I}_{r_i} \setminus I \\ \nabla_w V_i(z, \nu_{i,I}(z)) & \in & PC_{i,I} \end{array} \right\} \tag{3.32}$$

(the restriction $z \in \mathcal{Z} = X \times U$ is included in the definition of $Z_{i,I}$ since it is not included as a constraint in $\mathbb{P}_i(z)$). As before, we have to 'marry' a set of polytopes $Z_{i,I}$ for all i such that polytope P_i^Φ is active. Suppose, for each $i \in \mathcal{J}^\Phi$, polytope P_i^Φ is defined by

$$P_i^\Phi \triangleq \{(z, w) \mid M_i w \leq N_i z + p_i\} \tag{3.33}$$

for some M_i, N_i, p_i each having r_i rows. For each $i \in \mathcal{J}^\Phi$, let $P_i^{\mathcal{Z}}$ denote the polytope defined by

$$P_i^{\mathcal{Z}} \triangleq \{z \mid \exists\, w \text{ s.t. } (z, w) \in P_i^\Phi\} = \mathrm{Proj}_Z(P_i^\Phi) \tag{3.34}$$

For each $z \in P_i^{\mathcal{Z}}$, each $i \in \mathcal{J}^\Phi$, the set of active constraints for $\mathbb{P}_i(z)$ is

$$I_i^0(z) \triangleq \{j \in \mathcal{I}_{r_i} \mid M_i^j \nu_i(z) = N_i^j z + p_i^j\} \tag{3.35}$$

where M_i^j is the j^{th} row of M_i, N_i^j the j^{th} row of N_i, and p_i^j the j^{th} row of p_i. Also, for each $\bar{z} \in \mathcal{Z}$, let the polytope $Z_i(\bar{z}) \subseteq P_i^{\mathcal{Z}}$ be defined by

$$Z_i(\bar{z}) \triangleq Z_{i, I_i^0(\bar{z})} \tag{3.36}$$

where, for each index set $I \subseteq \mathcal{J}^\Phi$, $Z_{i,I}$ is defined by (3.32). The polytope $Z(\bar{z})$ that figures in the parametric solution of $\mathbb{P}_{\max}(z)$ is defined, for each $\bar{z} \in \mathcal{Z}$, by

$$Z(\bar{z}) \triangleq \cap\{Z_i(\bar{z}) \mid i \in S_\Phi^0(\bar{z})\} \tag{3.37}$$

Clearly $\bar{z} \in Z(\bar{z})$ and $Z(z') = Z(\bar{z})$ for all $z' \in Z(\bar{z})$. For each $\bar{z} \in \mathcal{Z}$, let the functions $J_{\bar{z}}^0(\cdot)$ and $\nu_{\bar{z}}(\cdot)$ be defined on $Z(\bar{z})$ by

$$J_{\bar{z}}^0(z) \triangleq J_i^0(z), \; \forall i \in S_\Phi^0(\bar{z}) \tag{3.38}$$

$$\nu_{\bar{z}}(z) \triangleq \nu_i(z), \; \forall i \in S_\Phi^0(\bar{z}) \tag{3.39}$$

The domain of each function is $Z(\bar{z})$; that the functions are well defined follows from Proposition 7, (3.35) and (3.36) which show that $\nu_i(z) = \nu_j(z)$ for all $i, j \in S_\Phi^0(\bar{z})$, all $z \in Z(\bar{z})$. Summarizing, we have:

Theorem 2 *Suppose $V : \Phi \to I\!R$ is continuous piecewise quadratic and strictly convex on a polytopic partition $\mathcal{P}^{\Phi} = \{P_i^{\Phi} \mid i \in \mathcal{J}^{\Phi}\}$ of the polytope Φ and that, for each $i \in \mathcal{J}^{\Phi}$, P_i^{Φ} has a non-empty interior. Then the value function $J^0(\cdot)$ is continuous piecewise quadratic and strictly convex on a polytopic partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of $\mathcal{Z} \triangleq \{z \mid \exists w \in W \text{ s.t. } (z,w) \in \Phi\} = \mathrm{Proj}_Z(\Phi)$. The maximizer $\nu(\cdot)$ is piecewise affine on $\mathcal{P}^{\mathcal{Z}}$. The polytopes $P_i^{\mathcal{Z}}$ are each of the form $Z(\bar{z})$ for some $\bar{z} \in \mathcal{Z}$; the value function and optimal control law satisfy $J^0(z) = J^0_{\bar{z}}(z)$ and $\nu(z) = \nu_{\bar{z}}(z)$ for all $z \in Z(\bar{z})$ and some $\bar{z} \in \mathcal{Z}$.*

The proof of this result follows from Propositions 3 and 4 and the discussion above. As stated above, in the absence of further restrictions, $\Phi = X \times U \times W$. However, in our use of this result, the cost function $V(\cdot)$ has the form $V(z,w) = \ell(z,w) + V^0(Fz + Gw)$ where $F \triangleq [A, B]$ and $V^0(x)$ may be known only on a compact subset $\mathcal{X}$ of $I\!R^n$; in this case $\Phi = \{(z,w) \in X \times U \times W \mid Fz + Gw \in \mathcal{X}\}$. If A is invertible (which we assume) or X is compact, then Φ is a (compact) polytope with a polytopic partition.

4 H_∞ control; no state constraints

In this section, we consider the H_∞ constrained optimal control problem when the only constraints are $u \in U$ and $w \in W$, i.e. $X = X_f = I\!R^n$. In this case, the dynamic programming equations (2.1) - (2.4) simplify and are replaced by the conventional dynamic programming equations:

$$V_j^0(x) = \min_{u \in U} \max_{w \in W} \{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\} \tag{4.1}$$

$$\kappa_j(x) = \arg\min_{u \in U} \max_{w \in W} \{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\} \tag{4.2}$$

with boundary condition

$$V_0^0(x) = V_f(x) \tag{4.3}$$

The domain X_j of V_j^0 now satisfies $X_j = I\!R^n$, for all $j \geq 0$ so the recursion equation (2.3) for X_j is not required. The recursion equations may be rewritten in the form

$$V_j^0(x) = \min_{u \in U} J_{j-1}^0(x,u) \tag{4.4}$$

$$J_{j-1}^0(z) \triangleq \max_{w \in W} \{\ell(z,w) + V_{j-1}^0(f(z,w))\} \tag{4.5}$$

$$\kappa_j(x) = \arg\min_{u \in U} J_{j-1}^0(x,u) \tag{4.6}$$

$$\nu_j(z) = \arg\max_{w \in W} \{\ell(z,w) + V_{j-1}^0(f(z,w))\} \tag{4.7}$$

where $z = (x, u)$. The important feature of this problem is that the constraint $u \in U$ in subproblem (4.4) has the same simple form as the constraint $w \in W$; this permits us to obtain stronger properties for the value functions $V_j^0(\cdot)$ and $J_j^0(\cdot)$, $j \geq 0$. The prototype problems for (4.4) and (4.5) are, respectively:

$$\mathbb{P}_{\min}(x) : \qquad V^0(x) = \min_u \{J(x, u) \mid u \in U\} \tag{4.8}$$

$$\mathbb{P}_{\max}(z) : \qquad J^0(z) = \max_w \{V(z, w) \mid w \in W\} \tag{4.9}$$

in which $V^0(\cdot)$ and $J(\cdot)$ replace, respectively, $V_j^0(\cdot)$ and $J_{j-1}^0(\cdot)$ in (4.4) and $J^0(\cdot)$ and $V(\cdot)$ replace, respectively, $J_{j-1}^0(\cdot)$ and $\ell(\cdot) + V_{j-1}(f(\cdot))$ in (4.5). Since $J(\cdot)$ is convex in x and $V(\cdot)$ is (under appropriate conditions) concave in w, their respective value functions have identical properties ($\min_u\{J(x, u) \mid u \in U\} = -\max_u\{-J(x, u) \mid u \in U\}$).

Proposition 9 *Suppose that $J(\cdot)$ in $\mathbb{P}_{\min}$ ($V(\cdot)$ in $\mathbb{P}_{\max}$) is continuously differentiable, strictly convex in u (strictly concave in w) and that U (W) is compact. Then the value function $V^0(\cdot)$ of $\mathbb{P}_{\min}$ ($J^0(\cdot)$ of $\mathbb{P}_{\max}$) is continuously differentiable.*

Proof: It is only necessary to consider $\mathbb{P}_{\max}$ and establish the continuous differentiability of $J^0(\cdot)$. Since W, being constant, is continuous in z, the continuity of the value function $J^0(\cdot)$ follows from the maximum theorem (e.g. Theorem 5.4.1 in [23]). Since the function $w \mapsto V(z, w)$ is strictly concave for all z, the maximizer $\nu(z)$ is unique (a singleton) for each z; by the same maximum theorem, $\nu(\cdot)$ is continuous. Since $V(\cdot)$ is continuously differentiable and W is compact, and the maximizer $\nu(z)$ is unique and continuous, it follows from the proof of Theorem 5.4.7 in [23] that the directional derivative of $J^0(\cdot)$ satisfies

$$dJ^0(z; h) = (\partial/\partial z)V(z, \nu(z))h \tag{4.10}$$

at any z, any direction h. Hence $J^0(\cdot)$ is Gateau differentiable at any $z \in \mathcal{Z}$ with Gateau derivative $G(z) = (\partial/\partial z)V(z, \nu(z))$. Since $G(\cdot)$ is continuous, $J^0(\cdot)$ is continuously (Frechet) differentiable in $\mathcal{Z}$ with derivative $(\partial/\partial z)J^0(z) = (\partial/\partial z)J(z, \nu(z))$ [27]. ■

Although the domain of the value functions $V_j^0(\cdot)$ is $\mathbb{R}^n$ and that of the value functions $J_j^0(\cdot)$ is $\mathbb{R}^{n+m}$, we restrict attention in this section to polytopic subsets of these domains. With this caveat, we now show that there exists a $\gamma > 0$ such that $V(\cdot)$ in $\mathbb{P}_{\max}$, which represents $V^0_{j-1}(\cdot)$ in (4.5) and therefore has the form $V(z,w) = \ell(z,w) + V^0(Fz + Gw)$ where $F \triangleq [A, B]$, is strictly concave in w.

Proposition 10 *Let $\mathcal{X}$ be a polytope in $I\!R^n$ containing the origin in its interior. Suppose $V(\cdot)$ is defined by $V(z,w) \triangleq \ell(z,w)+V^0(Fz+Gw)$ where $V^0(\cdot)$ is continuously differentiable and continuous piecewise quadratic on a polyhedral partition $\mathcal{P}^{\mathcal{X}} = \{P_i^{\mathcal{X}} \mid i \in \mathcal{J}^{\mathcal{X}}\}$ of $\mathcal{X}$. Then $V(\cdot)$ is continuously differentiable and continuous piecewise quadratic on a polyhedral partition $\mathcal{P}^{\Phi} = \{P_i^{\Phi} \mid i \in \mathcal{J}^{\Phi}\}$ of the polyhedron $\Phi \triangleq \{(z,w) \in I\!R^n \times U \times W \mid Fz + Gw \in \mathcal{X}\}$ and there exists a $\gamma^* > 0$ such that $V(\cdot)$ is strictly concave in w for each z in $\mathcal{Z} \triangleq \mathrm{Proj}_Z\Phi$ and all $\gamma \geq \gamma^*$.*

Proof: The continuous differentiability of $V(\cdot)$ follows from the continuous differentiability of $\ell(\cdot)$ and $V^0(\cdot)$. Take any two points w_1, w_2 in W. For all $\lambda \in [0,1]$, let $w_\lambda \triangleq w_1 + \lambda(w_2 - w_1)$, and, for each $z \in \mathcal{Z}$, let the real valued function $\phi(\cdot)$ be defined on $[0,1]$ by $\phi(\lambda) \triangleq V(z, w_\lambda)$. Suppose that $V^0(x) = (1/2)x'Q_i x + q_i' x + r_i$ in $P_i^{\mathcal{X}}$ (for each $i \in \mathcal{J}^{\mathcal{X}}$). Then

$$\begin{aligned} V(z,w) &= (1/2)(Fz+Gw)'Q_i(Fz+Gw) + q_i'(Fz+Gw) + r_i + \ell(z,w) \\ &= -(1/2)w'(\gamma^2 I - G'Q_iG)w + b_i'w + c_i \end{aligned}$$

on the polyhedron $P_i^{\Phi} = \{(z,w) \in I\!R^n \times U \times W \mid Fz + Gw \in P_i^{\mathcal{X}}\}$, where b_i and c_i depend on z. For any $\varepsilon > 0$, there exists a $\gamma^* > 0$ such that $C_i \triangleq \gamma^2 I - G'Q_iG \geq \varepsilon I$ for all $\gamma \geq \gamma^*$, all $i \in \mathcal{J}^{\mathcal{X}}$. The function $\phi(\cdot)$ is continuously differentiable and satisfies:

$$\begin{aligned} \phi(\lambda) &= -(1/2)(h'C_ih)\lambda^2 + b_i\lambda + c_i \\ \phi'(\lambda) &= -(h'C_ih)\lambda + b_i \end{aligned}$$

for all $\lambda \in [0,1]$ such that $Fz + Gw_\lambda \in P_i^{\Phi}$. Since $\phi'(\cdot)$ is continuous, $\phi'(\cdot)$ is strictly decreasing if $\gamma \geq \gamma^*$. It follows, by a trivial modification to the proof of Theorem 4.4 in [28], that $\phi(\lambda) > \phi(0) + \lambda(\phi(1) - \phi(0))$ for all $\lambda \in (0,1)$ which establishes the strict concavity of $\phi(\cdot)$ and, hence, of $w \mapsto V(z,w)$ if $\gamma \geq \gamma^*$. That $V(\cdot)$ is piecewise quadratic on a polyhedral partition of $\mathcal{P}^{\Phi}$ follows, with minor amendments, from the proofs of Proposition 8 and Theorem 2. ■

We can now establish the main result of this section, characterization of the solution to the constrained H_∞ problem when $X = X_f = I\!R^n$. We characterize the value functions V_j^0 on polytopic subsets X_j of the true domain $I\!R^n$ by assuming that the terminal cost function $V_f(\cdot)$ is known only in a polytopic subset X_0 of $I\!R^n$.

Theorem 3 *Suppose $V_f(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polytopic partition $\mathcal{P}_0^X$ of a polytope $X_0 \subset I\!R^n$. Then, there exists a $\gamma > 0$ such that, for each $j \geq 0$, there exists a polyhedron X_j on which the value function $V_j^0(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polyhedral partition $\mathcal{P}_j^X$ of X_j, and the optimal control law $\kappa_j(\cdot)$ is continuous and piecewise affine on the same polyhedral partition $\mathcal{P}_j^X$ of X_j.*

Proof: Suppose $V_{j-1}^0(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polyhedral partition $\mathcal{P}_{j-1}^X$ of a polyhedron X_{j-1} if $\gamma \geq \gamma_{j-1}$. Then, by Proposition 10, there exists a $\gamma_j \geq \gamma_{j-1}$ such that $(z, w) \mapsto \ell(z, w) + V_{j-1}^0(f(z, w))$ is strictly concave in w, continuously differentiable and continuous piecewise quadratic on a polyhedral partition $\mathcal{P}_{j-1}^\Phi$ of a polyhedron $\Phi_{j-1} = \{(z, w) \in I\!R^n \times U \times W \mid Fz + Gw \in X_{j-1}\}$. By Proposition 9, the value function $J_{j-1}^0(\cdot)$ is then continuously differentiable and, by Theorem 2, $J_{j-1}^0(\cdot)$ is continuous piecewise quadratic and strictly convex on a polyhedral partition $\mathcal{P}_{j-1}^Z$ of a polyhedron $Z_{j-1} = \mathrm{Proj}_Z \Phi_{j-1}$ (and the disturbance law $\nu_{j-1}(\cdot)$ is continuous and piecewise affine on the same polytopic partition). Then, by Proposition 9, $V_j^0(\cdot)$ is continuously differentiable and, by Theorem 1, $V_j^0(\cdot)$ is strictly convex and continuous piecewise quadratic on a polyhedral partition $\mathcal{P}_j^X$ of a polyhedron X_j (and the optimal control law $\nu_{j-1}(\cdot)$ is continuous and piecewise affine on the same polyhedral partition). ■

5 H_∞ control; state and control constraints

In this section, we consider the H_∞ constrained optimal control problem when the constraints are $u \in U$, $w \in W$, $x \in X$ and the terminal constraint $x_N \in X_f$. The dynamic programming solution of the H_∞ problem requires the repeated solution of the two prototype problems $\mathbb{P}_{\min}(x)$ and $\mathbb{P}_{\max}(z)$ defined in (3.1) and (3.2) which we rewrite in the form:

$$\mathbb{P}_{\min}(x): \qquad V^0(x) = \min_u \{J(x,u) \mid u \in \mathcal{U}(x)\} \tag{5.1}$$

$$\mathbb{P}_{\max}(z): \qquad J^0(z) = \max_w \{V(z,w) \mid w \in W\} \tag{5.2}$$

where the set-valued function $\mathcal{U}(\cdot)$ is defined, for all $x \in \mathcal{X} \triangleq \mathrm{Proj}_X(Z)$, by

$$\mathcal{U}(x) \triangleq \{u \mid (x,u) \in Z\} \tag{5.3}$$

The presence of state constraints complicates the solution of the H_∞ problem considerably. The extra complexity arises in the solution of $\mathbb{P}_{\min}(x)$ since the control constraint $u \in \mathcal{U}(x)$ is now dependent on the parameter x in contrast to the simple constraint $u \in U$ when no state constraints are present. The dependency of the constraint on x can cause the gradient of the value function $V^0(\cdot)$ in problem $\mathbb{P}_{\min}(x)$ to be discontinuous even if the function $J(\cdot)$ being minimized is continuously differentiable; Proposition 9 is no longer necessarily true for problem $\mathbb{P}_{\min}(x)$. However, there do exist conditions under which this result is true.

5.1 Particular case

Assume that $J(\cdot)$ in (3.1) is continuously differentiable and continuous piecewise quadratic on a polytopic partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}}, i \in \mathcal{J}^{\mathcal{Z}}\}$ of $\mathcal{Z}$. We show below, despite the fact that the constraint set $\mathcal{U}(x)$ now depends on the parameter x, that the value function $V^0(\cdot)$ for $\mathbb{P}_{\min}$ is, under certain further assumptions, continuously differentiable in $\mathcal{X} \triangleq \mathrm{Proj}_X(\mathcal{Z})$. To do this, we first consider, as in §3.1, the simpler problems $\mathbb{P}_i(z)$, $i \in \mathcal{J}^{\mathcal{Z}}$, defined by (3.8). For each i, problem $\mathbb{P}_i(z)$ is a quadratic program, with a value function $V_i^0(\cdot)$ that is continuous piecewise quadratic on a polytopic partition of the polytope $P_i^{\mathcal{X}} \triangleq \mathrm{Proj}_X(P_i^{\mathcal{Z}})$, and may be written in the form

$$\mathbb{P}_i(x): \qquad V_i^0(x) = \min_u \{J_i(x,u) \mid x \in \mathcal{U}_i(x)\} \tag{5.4}$$

where

$$\mathcal{U}_i(x) \triangleq \{u \mid (x,u) \in P_i^{\mathcal{Z}}\} = \{u \mid M_i u \leq N_i x + p_i\} \tag{5.5}$$

It is known (see Proposition 4 and Theorem 1) that the value function $V_i^0(\cdot)$ is continuous piecewise quadratic, being quadratic on polytopes $X_{i,I}$, each polytope characterized by a set $I \subseteq \mathcal{I}_{r_i}$ of active constraints, where r_i is the number of rows of M_i; the sets $X_{i,I}$, $I \subseteq \mathcal{I}_{r_i}$ (excluding sets with no interior) constitute a polytopic partition of the polytope $P_i^{\mathcal{X}}$. We require the following result which is proved in the appendix.

Proposition 11 *Suppose (i), $J_i(\cdot)$ is continuously differentiable, (ii) $P_i^{\mathcal{Z}}$ has an interior, and, (iii) for any two adjacent polytopes, X_{i,I_1} and X_{i,I_2} say, in the polytopic partition of $P_i^{\mathcal{X}}$, either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. Then, $V_i^0(\cdot)$ is continuously differentiable in $P_i^{\mathcal{X}}$.*

We establish next the continuous differentiability of the value function $V^0(\cdot)$ for $\mathbb{P}_{\min}$.

Theorem 4 *Suppose that $J(\cdot)$ in (3.1) is continuously differentiable and continuous piecewise quadratic on a polytopic partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}}, i \in \mathcal{J}^{\mathcal{Z}}\}$ of $\mathcal{Z}$ and that hypotheses (ii) and (iii) of Proposition 11 are satisfied (hypothesis (i) is satisfied automatically) for each problem $\mathbb{P}_i$, $i \in \mathcal{J}^{\mathcal{Z}}$. Then $V^0(\cdot)$, the value function for $\mathbb{P}_{min}$, is continuously differentiable in $\mathcal{X}$.*

Proof: It follows from Theorem 1, that, for each $i \in \mathcal{J}^{\mathcal{Z}}$, $V^0(x) = V_i^0(x)$ for all x in the interior of each polytope $X_{i,I}$ in the polytopic partition of $P_i^{\mathcal{X}}$. It follows from Proposition 11, that $V^0(\cdot)$ is continuously differentiable in $P_i^{\mathcal{X}}$ for each $i \in \mathcal{J}^{\mathcal{Z}}$. Consider next a point $\bar{x}$ on the boundary between two polytopes $P_i^{\mathcal{X}}$ and $P_j^{\mathcal{X}}$; clearly i and j both lie in $S_{\mathcal{Z}}^0(\bar{x})$ and, from Theorem 1, $V^0(x) = V_i^0(x) = V_j^0(x)$ for all $x \in X(\bar{x})$, so that $V^0(\cdot)$ is continuously differentiable in $X(\bar{x})$ and, hence, on all boundaries between polytopes in the polytopic partition of $\mathcal{X}$. ∎

Theorem 5 *Suppose $V_f(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polytopic partition $\mathcal{P}_0^X$ of a polytope $X_0 \subset \mathbb{R}^n$. Then, there exists a $\gamma > 0$ such that, for each $j \geq 0$, there exists a polyhedron X_j on which the value function $V_j^0(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polytopic partition $\mathcal{P}_j^X$ of X_j, and the optimal control law $\kappa_j(\cdot)$ is continuous and piecewise affine on the same polytopic partition $\mathcal{P}_j^X$ of X_j.*

Proof: Suppose $V_{j-1}^0(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polytopic partition $\mathcal{P}_{j-1}^X$ of a polytope X_{j-1} if $\gamma \geq \gamma_{j-1}$. Then, by Proposition 10, there exists a $\gamma_j \geq \gamma_{j-1}$ such that $(z, w) \mapsto \ell(z, w) + V_{j-1}^0(f(z, w))$ is strictly concave in w, continuously differentiable and continuous piecewise quadratic on a polytopic partition $\mathcal{P}_{j-1}^{\Phi}$ of a polytope $\Phi_{j-1} = \{(z, w) \in X \times U \times W \mid Fz + Gw \in X_{j-1}\}$. By Proposition 9, the value function $J_{j-1}^0(\cdot)$ is then continuously differentiable and, by Theorem 2, $J^0_{j-1}(\cdot)$ is continuous piecewise quadratic and strictly convex on a polytopic partition $\mathcal{P}^Z_{j-1}$ of a polytope $Z_{j-1} = \mathrm{Proj}_Z \Phi_{j-1}$ (and the disturbance law $\nu_{j-1}(\cdot)$ is continuous and piecewise affine on the same polytopic partition). Then, by Proposition 9, $V^0_j(\cdot)$ is continuously differentiable and, by Theorem 1, $V^0_j(\cdot)$ is strictly convex and continuous piecewise quadratic on a polytopic partition $\mathcal{P}^X_j$ of a polytope X_j (and the optimal control law $\nu_{j-1}(\cdot)$ is continuous and piecewise affine on the same polytopic partition). ■

5.2 General case

A simple characterization for the solution of the H_∞ problem with control and state constraints does not appear possible when the simplifying assumption of §5.1 is not made. Without this assumption, the value function $V^0(\cdot)$ for the minimization problem $\mathbb{P}_{\min}$ is not necessarily continuously differentiable at the boundary between polytopes in the polytopic partition of $\mathcal{X}$. Consequently, the objective function $V(\cdot)$ (which has the form $V = \ell + V^0$) in the maximization problem is not necessarily concave, no matter how large γ is chosen. The resultant cost function $J(\cdot)$ in the minimization problem is then piecewise max-quadratic, i.e. it is continuous and equal to the maximum of a finite number of quadratics in each polytope in a polytopic partition of its domain. It does not appear possible to obtain a simple characterization for a problem with this structure.

5.3 Illustrative example

The partial value functions and optimal control laws can be computed by solving the max and min subproblems associated (for each subproblem) with each set of potentially active constraints and each set of potentially active polytopes. Since the number of these sets is combinatorial, a better procedure, employed in our computations, is to select a state-control pair $z = (x, u)$ in the max subproblem, determine the active constraint set J or the active index set of polytopes s, and then compute the corresponding disturbance law $\nu(\cdot)$ and the region Z in which this control law is optimal, using Theorem 2. The procedure is then repeated for a new value of z not lying in the union of the sets Z already computed. Once the max subproblem computations are complete, a similar procedure is applied to the min subproblem using Theorem 1.

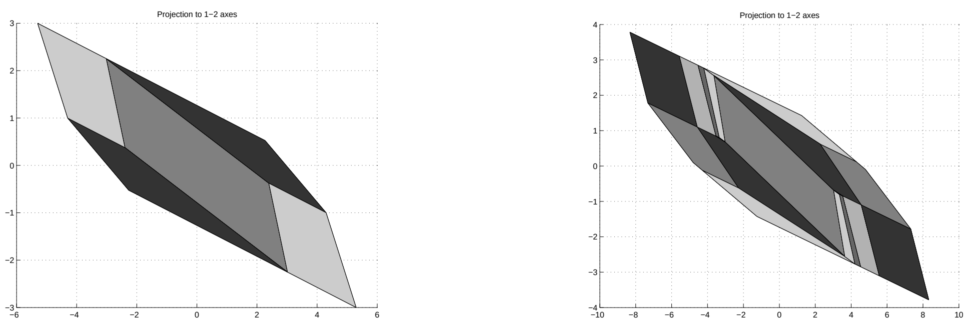
Our numerical example is optimal min-max control of a constrained second order system defined by:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w \tag{5.6}$$

The state constraints are $x \in X := \{x \mid |x|_\infty \leq 10\}$. The control constraint is $u \in U := \{u \mid |u| \leq 1\}$. The disturbance is bounded: $w \in W := \{w \mid |w|_\infty \leq 0.1\}$ The path cost function is quadratic with $Q = 10I$, $R = 1$ and $\gamma = 100$. The terminal cost $V_f(x)$ is quadratic $(1/2)x'P_f x$ with

$$P_f = \begin{bmatrix} 20.6143 & 5.9244 \\ 5.9244 & 14.2329 \end{bmatrix}$$

The terminal constraint set X_f is defined by the 4 inequalities: $(-0.9849\ -0.3155)x \leq 2.1526$; $(0.9489\ 0.3155)x \leq 2.1526$; $(0.4369\ 0.8995)x \leq 0.7079$ and $(-0.4369\ -0.8995)x \leq 0.7079$.



(a) Regions of x space, $N = 1$

(b) Regions of x space, $N = 2$

Figure 3: 2D Example

The polytopic regions for $N = 1$ and $N = 2$ are shown in Figure 3. At $N = 2$, the state space is partitioned into 17 polytopes in each of which the optimal control law is piecewise affine; the state control space $\mathcal{Z}$ is partitioned into 5 polytopes in each of which the optimal disturbance law is piecewise affine.

6 H_∞ receding horizon control

6.1 Introduction

Since we make use, in this section, of the solution for infinite horizon, linear unconstrained H_∞ problem, we assume, in the sequel, that (A, B) is stabilizable and that (C, A, B) has no zeros on the unit circle where $Q = C'C$. Since Q is assumed to be positive definite, (C, A) is detectable. These conditions, and the fact that R is assumed positive definite, ensure that the conditions assumed in [29], Appendix B, are satisfied for the full information case. Hence there exists a $\tilde{\gamma} > 0$ such that a positive definite solution P_f to the associated (generalized) H_∞ algebraic Riccati equation for all $\gamma > \tilde{\gamma}$ and associated optimal control and disturbance laws $u = K_u x$ and $w = K_w x$ respectively. It is shown in [29] that, under these assumptions, the state matrices $A_f \triangleq A + BK_u$ and $A_c \triangleq A + BK_u + GK_w$ are both stable.

The terminal cost function $V_f(\cdot)$ for the constrained H_∞ control problem is defined by

$$V_f(x) = (1/2)|x|^2_{P_f}. \tag{6.1}$$

and satisfies

$$V_f(A_c x) - V_f(x) + \ell(x, K_u x, K_w x) = 0. \tag{6.2}$$

The terminal constraint set X_f is chosen to be a disturbance invariant set (if it exists) for the system $x^+ = A_f x + Gw$, $A_f \triangleq A + BK_u$. Any disturbance invariant set X_f satisfies

$$f(x, K_u x, W) \subseteq X_f \ \forall\ x \in X_f \tag{6.3}$$

We assume that the set W is sufficiently small, and that γ is sufficiently large, to ensure the existence of a disturbance invariant set X_f which satisfies

$$X_f \subseteq X, \quad K_u X_f \subseteq U, \quad K_w X_f \subseteq W. \tag{6.4}$$

That the last condition in (6.4) can be satisfied follows from the fact [29] that $K_w \to 0$ as $\gamma \to \infty$. A suitable X_f may be computed as follows: if $W = \{w \mid C_w w \leq c_w\}$, choose γ such that $\tilde{X} = \{x \mid C_w K_w x \leq c_w\}$ is reasonably large; clearly $K_w \tilde{X} \subseteq W$. Next, choose X_f to be a disturbance invariant set for $x^+ = A_f x + Gw$ satisfying $X_f \subseteq X \cap \tilde{X}$ and $K_u X_f \subseteq U$.

Since there does not exist a disturbance w that can steer the system outside X_f given an initial state in X_f, the optimal policy for w in X_f is $w = K_w x$. The closed loop system $x^+ = A_c x$, $A_c \triangleq A + BK_u + GK_w$, is exponentially stable and the controller $u = K_u x$ maintains the state in X_f if the initial state is in X_f. We observe that the solution of the infinite horizon constrained H_∞ problem (defined by (2.9) with $N = \infty$) satisfies:

$$V^0_\infty(x) = V_f(x), \quad \kappa_\infty(x) = K_u x, \ \forall x \in X_f \tag{6.5}$$

since, by (6.4), the control constraints are satisfied everywhere in X_f so that the solutions of the constrained and unconstrained problems coincide.

6.2 H_∞ control: control constraints

Since this problem, as stated in §4, has no terminal constraint and since $V_f(\cdot)$ defined above is a local rather than a global Control Lyapunov Function ($V_f(\cdot)$ is valid in X_f), standard stability results [30, 31] (that enforce the terminal constraint) cannot be employed. However, it is possible to determine a domain of attraction for the H_∞ controller characterized in §4. Consider the following dynamic programming recursion:

$$V_j^0(x) = \min_{u\in U}\max_{w\in W}\{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\} \tag{6.6}$$

$$\kappa_j(x) = \arg\min_{u\in U}\max_{w\in W}\{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\} \tag{6.7}$$

$$X_j^* = \{x \mid f(x,\kappa_j(x),W) \subseteq X_{j-1}^*\} \tag{6.8}$$

with boundary condition

$$V_0^0(x) = V_f(x), \quad X_0^* = X_f \tag{6.9}$$

This is identical to the recursion (4.4)–(4.6) except for the inclusion of the recursion (6.8) that yields the sets X_j^*, $j \geq 0$. Whereas the domain of the value function $V_j^0(\cdot)$ is $I\!R^n$ for all $j \geq 0$, the importance of the sets X_j^* derives from Proposition 12 below.

Proposition 12 *For every integer $j \geq 0$, every $x \in X_j^*$:*

$$V_j^0(x) = V_\infty^0(x) \tag{6.10}$$

$$\kappa_j(x) = \kappa_\infty(x) \tag{6.11}$$

Proof: Suppose, for some integer j, $V_{j-1}^0(\cdot) = V_\infty^0(\cdot)$ on X_{j-1}^*. Then, by (6.6),

$$V_j^0(x) = \min_{u\in U}\max_{w\in W}\{\ell(x,u,w) + V_\infty^0(f(x,u,w))\} = V_\infty^0(x)$$

for all $x \in X_j^*$. Since $V_f(x) = V_\infty^0(x)$ for all $x \in X_0 = X_f$, the desired result follows by induction. ∎

Hence, the solution to the finite horizon H_∞ problem in §4 is also the solution to the infinite horizon problem (in the restricted sets X_j^*) provided the terminal cost $V_f(\cdot)$ is chosen as described above. A practical consequence of this result is that, in computing the value function $V_j^0(\cdot)$ (and $\kappa_j(\cdot)$), it is only necessary to consider those states lying in $X_j^* \setminus X_{j-1}^*$ (since $V_j^0(x) = V_{j-1}^0(x) = V_\infty(x)$ at all $x \in X_{j-1}^*$).

Definition 4 *A set X is robust control invariant for $x^+ = f(x,u,w)$ if, for every $x \in X$, there exists a $u \in U$ such that $f(x,u,W) \subseteq X$.*

It follows from (6.3) that the set X_f is robust control invariant.

Theorem 6 *(i) The sets X_j^* are each robust control invariant and are non-decreasing (satisfy $X_j^* \subseteq X_{j+1}^*$ for all $j \geq 0$). (ii) For any $N \geq 0$, the set X_f is finite-time attractive with a domain of attraction X_N^* for the closed-loop system $x^+ = f(x, \kappa_\infty(x), w)$. (iii) Suppose that, for some finite integer $j \leq N$, X_f lies in the interior of X_j^*; then X_f is robustly stable for the system $x^+ = f(x, \kappa_\infty(x), w)$.*

Proof: (i) Assume X_{j-1}^* is robust control invariant. It follows from (6.8) that $X_{j-1}^* \subseteq X_j^*$ so that X_j^* is robust control invariant. Since X_f is robust control invariant, the desired result follows by induction. (ii) By construction, for any integer j, and state $x \in X_j^*$ is robustly steered into X_{j-1}^* by the admissible control $\kappa_j(x) = \kappa_\infty(x)$. Hence any state $x \in X_N^*$ is robustly steered into X_f in N steps; the controller $u = K_u x$ then keeps the state in X_f, so that X_f is robustly finite-time attractive with a domain of attraction X_N^* for the system $x^+ = f(x, \kappa_\infty(x), w)$. (iii) For any $x \in I\!R^n$ let $|x|^H \triangleq d(x, X_f)$ and for any infinite sequence $\{x(i)\}$ in $I\!R^n$ let $|\{x(i)\}|_\infty^H \triangleq \sup_{i \geq 0} d(x(i), X_f)$. From (ii), the controller $\kappa_\infty(\cdot)$ steers any $x \in X_j^*$ into X_f in no more than j steps and, thereafter, keeps the state in X_f. Hence, with $x(i) \triangleq \phi(i; x, \pi_\infty, \{w(i)\})$, $\mathbf{w} \triangleq \{w(0), w(1), \ldots, w(j-1)\}$, let $\theta : X_j^* \times W^j \to I\!R$ be defined by

$$\theta(x, \mathbf{w}) \triangleq |x(\cdot)|_\infty^H = \max_i \{\phi(i; x, \pi_\infty, \mathbf{w}\} \mid i \in \{0, 1, \ldots, j-1\}\}.$$

The control law $\kappa_\infty(\cdot)$ is continuous since it is equal to $\kappa_j(\cdot)$ in X_j^*; thus $\theta(\cdot)$ is continuous and, hence, uniformly continuous in $X_j^* \times W^j$. Since $\theta(x, \mathbf{w}) = 0$ for all $x \in X_f \subset X_j^*$, all $\mathbf{w} \in W^j$, uniform continuity of $\theta(\cdot)$ implies that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\theta(x, \mathbf{w}) < \varepsilon$ ($x(i) \in B_\varepsilon(X_f)$ for all $i \geq 0$) for all $(x, \mathbf{w}) \in X_j^* \times W^j$ satisfying $|x|^H < \delta$ ($x \in B_\varepsilon(X_f)$). This establishes robust stability of X_f. ■

The disadvantage of this approach is that the sets X_j^* are obviously subsets of $I\!R^n$, the domain of the value functions $V_j^0(\cdot)$; the sets X_j^* are not necessarily convex.

6.3 H_∞ control: state and control constraints

Consider the receding horizon controller $u = \kappa_N(x)$. If the terminal conditions and assumptions stated above in §6.1 are adopted, then

C1: X_f is robust control invariant for $x^+ = (A + BK_u)x + Gw$, $X_f \subseteq X$, $K_u X_f \subseteq U$, $K_w X_f \subseteq W$.

C2: $\min_{u \in U} \max_{w \in W} \{[\overset{*}{V}_f + \ell](x, u, w) \mid f(x, u, w) \in X_f\} \leq 0$ for all $x \in X_f$.

In C2, $\overset{*}{V}_f(x, u, w) \triangleq V_f(f(x, u, w)) - V_f(x)$. If the recursive dynamic programming equations in (2.1) – (2.3) are employed, we obtain, by a minor modification of the results in [22], §3.3.1, the following results:

(i:) X_i is robust control invariant for all $i \in \{1, \dots, N\}$
(ii:) X_N is robust invariant for $x^+ = f(x, \kappa_N(x), w)$
(iii:) $V_i^0(x) \leq V_{i-1}^0(x)\ \forall x \in X_{i-1},\ i \in \{1, \dots, N\}$
(iv:) $V_N^0(x) \leq V_f(x)\ \forall\ x \in X_f$.
(v:) The value function satisfies:

$$[(\overset{*}{V_N^0} + \ell) \leq (V_N^0 - V_{N-1}^0)](f(x, \kappa_N(x), w) \leq 0$$

for all $(x, w) \in X_N \times W$. Property (iii) is the monotonicity property of the value function for the *constrained*, linear, uncertain system (1.1) with cost (1.5). Let $V_N(x, \pi, \mathbf{w})$ denote the cost if player u uses the control law $u = \kappa_N(x)$ and the adversary w uses an arbitrary admissible disturbance sequence $\mathbf{w}$. Then, for any ℓ_2 disturbance sequence

$$\begin{aligned} V_N(x, \pi, \mathbf{w}) &= \sum_{i=0}^{N-1} |y(i)|^2 - (\gamma^2/2)|w(i)|^2 + V_f(x(N)) \\ &= \sum_{i=0}^{\infty} |y(i)|^2 - (\gamma^2/2)|w(i)|^2 \leq V_N^0(x) \qquad (6.12) \end{aligned}$$

since $\mathbf{w}$ is not optimal; here $y(i) = Hz(i)$, $z(i) = (x(i), u(i)) = (x(i), \kappa_N(x(i))$ and $x(i)$ is the solution of (1.1) due to initial state x, control strategy π and disturbance sequence $\mathbf{w}$; we make use of the fact that $\kappa_N(x) = \kappa_f(x)$ for all $x \in X_f$. It follows that

$$\sum_{i=0}^{\infty} |y(i)|^2 \leq (\gamma^2/2) \sum_{i=0}^{\infty} |w(i)|^2 + V_N^0(x) \qquad (6.13)$$

which is the finite gain property. Next, if the disturbance is identically zero,

$$[(\overset{*}{V_N^0} + \ell) \leq (V_N^0 - V_{N-1}^0)](f(x, \kappa_N(x), 0)) \leq 0$$

so that

$$V_N^0(f(x, \kappa_N(x), 0)) - V_N^0(x) \leq -\ell(x, \kappa_N(x), 0) \leq -(1/2)x'Qx$$

for all $x \in X_N$ so that the origin is exponentially stable with a region of attraction X_N. Summarizing we have

Theorem 7 *The receding horizon controller $u = \kappa_N(x)$ has the following properties. The controlled system has the finite ℓ_2 gain property* (6.13) *for every initial state in the interior of X_N and, if the disturbance is identically zero, the origin is exponentially stable with a region of attraction X_N.*

If the disturbance satisfies $w \in W(z)$, where, as before, $z = (x, u)$ and W is such that $w \in W(z)$ implies $|w| \leq \delta|z|$, for some $\delta > 0$ (this W models some parametric uncertainties), then

$$\ell(z, w) = (1/2)|z|^2_{HH'} - (\gamma^2/2)|w|^2 \geq (c/2)|z|^2 \geq (c/2)|x|^2$$

for all z, w, some $c > 0$, provided that $\delta < (1/\gamma)$. With this form of bounded disturbance, the origin is robustly, exponentially stable (the state converges to the origin exponentially fast despite the disturbance) if, of course, $\delta < (1/\gamma)$.

We note, in passing, that we can simplify the dynamic programming recursion, as in §6.2, by replacing (2.3) by

$$X_j^* = X \cap \{x \mid f(x, \kappa_j(x), W) \subseteq X_{j-1}^*\}$$

and the boundary conditions by

$$V_0^0(x) = V_f(x), \quad X_0^* = X_f$$

However, in this case, (2.1) remains a constrained optimization problem because of the state constraint, so the advantage of using this formulation is not so clear cut. As before, $X_j^* \subseteq X_j$ for each j which introduces conservatism. However, the sets X_j^* are less complex than the corresponding sets X_j.

7 Conclusion

We have shown (in §4) how the solution to the constrained H_∞ problem may be characterized when the system is linear, the cost quadratic and the constraints polytopic if no state and/or terminad constraints are present. This characterization required the solution to a parametric program in which the constraints are polytopic and the cost piecewise quadratic (rather than quadratic). A novel solution to this problem is presented in §3. A characterization of the solution to the constrained H_∞ problem when state constraints are present under special (and restrictive) conditions is presented in §5; characterization of the solution in the general case appears to be difficult. Stability properties of the resultant H_∞ controlled system are briefly discussed in §6.

APPENDICES

A Proof of Lemma 1, §3.1

We restate Lemma 1:

Lemma 1 (Clarke) *Suppose $\mathcal{Z}$ is a polytope in $I\!R^n \times I\!R^m$ and let $\mathcal{X}$ denote its projection on $I\!R^n$ ($\mathcal{X} = \{x \mid \exists u \in I\!R^m$ such that $(x,u) \in \mathcal{Z}\}$). Let $\mathcal{U}(x) \triangleq \{u \mid (x,u) \in \mathcal{Z}\}$. Then there exists a $K > 0$ such that, for all $x, x' \in \mathcal{X}$, for all $u \in \mathcal{U}(x)$, $d(u, \mathcal{U}(x')) \leq K|x' - x|$ (there exists a $u' \in \mathcal{U}(x')$ such that $|u' - u| \leq K|x' - x|$).*

Proof: The polytope $\mathcal{Z}$ is defined by

$$\mathcal{Z} \triangleq \{z = (x,u) \mid Mu \leq Nx + p,\ Lx \leq l\}.$$

where the second set of inequalities is introduced to ensure that no row of M is zero ($Lx \leq l$ defines the right hand boundary of $\mathcal{Z}$ in Figure 1). Let r denote the row-dimension of M, s the row dimension of L, let $I \triangleq \{1, \ldots, r\}$ and $J \triangleq \{1, \ldots, s\}$. For all $(x,u) \in \mathcal{Z}$, $\mathcal{U}(x) = \{u \mid Mu \leq Nx + p\}$ (since $(x,u) \in \mathcal{Z} \implies Lx \leq l$). For each $x \in \mathcal{X}$, $\mathcal{U}(x)$ is a polytope in $I\!R^m$. For each $(x,u) \in \mathcal{Z}$, let $\psi(x,u) \triangleq \max\{M^i u - N^i x - p^i \mid i \in I\}$ and let $\psi^+(x,u) \triangleq \max\{0, \psi(x,u)$ where M^i, N^i, p^i denote the ith row, respectively, of M, N and p. Then, for all $x \in \mathcal{X}$, $u \in \mathcal{U}(x)$ $((x,u) \in \mathcal{Z})$ if and only if $\psi(z) = 0$). Let $I^0(z) \triangleq \{i \in I \mid M^i u - N^i x - p^i = \psi(z)\}$ index the (most) active constraints. The associated set of (most) active gradients (with respect to u) is $\{g_i \mid i \in I^0(z)\}$ where, for each i, $g_i = (M^i)'$. Because $\mathcal{U}(x)$ is a polytope, for each $x \in \mathcal{X}$, the set of active gradients is positively linear independent ($0 \notin \text{co}\{g_i(z) \mid i \in I^0(z)\}$ for all z such that $\psi(z) > 0$. The proximal subgradient of $u \mapsto \psi(x, u)$ is:

$$\delta_p(x, u)\psi(x, u) = \text{co}\{g_i \mid i \in I^0(z)\}.$$

and the directional derivative $d\psi(z; h)$ of $\psi(\cdot)$ at z in direction h is $\max\{\langle g_i, h\rangle \mid i \in I^0(z)\}$; the positive linear independence condition ensures that, at each z, there exists a direction h along which $\psi(z)$ can be decreased. In fact, there exists a $\delta > 0$ such that, if $z \in \mathcal{Z}$ and $\psi(z) > 0$ ($z \notin \mathcal{Z}$), then $\zeta \in \delta_p(z)$ implies $|\zeta| \geq \delta$. So, by Theorem 3.1 in [32], for all $(x, u) \in \mathcal{Z}$, all $x' \in \mathcal{X}$, $d(u, \mathcal{U}(x')) \leq \psi(x', u)/\delta \leq (c/\delta)|x - x'|$ ($c \triangleq \max\{|N^j| \mid j \in J\}$) since $\psi(x', u) \leq \psi(x, u) + \max\{N^j(x - x') \mid j \in J\} \leq c|x - x'|\}$. This proves the lemma with $K \triangleq c/\delta$. ■

B Continuous differentiability of the value function of a parametric quadratic program

See [22] and [8] for related results. We consider the standard parametric quadratic program:

$$\mathbb{P}(x) : \quad V^0(x) = \min_u \{V(x, u) \mid (x, u) \in \mathcal{Z}\}. \tag{B.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^l$ and $\mathcal{Z}$ is a polytope with a non-empty interior. We assume

A1: The cost function $V(\cdot)$ is strictly convex and continuously differentiable. The constraint $(x, u) \in \mathcal{Z}$ imposes an implicit constraint on $u \in \mathcal{U}(x)$ where the set-valued function $\mathcal{U}(\cdot)$ is defined by

$$\mathcal{U}(x) = \{u \mid (x, u) \in \mathcal{Z}\} = \{u \mid Mu \leq Nx + p\} \tag{B.2}$$

so that $\mathbb{P}(x)$ may be written in the form

$$\mathbb{P}(x) : \quad V^0(x) = \min_u \{V(x, u) \mid u \in \mathcal{U}(x)\} \tag{B.3}$$

The (unique) solution of $\mathbb{P}(x)$, for each $x \in \mathcal{X}$, is

$$u^0(x) = \arg\min_u \{V(x,u) \mid u \in \mathcal{U}(x)\} \tag{B.4}$$

The domain of $V^0(\cdot)$, $u(\cdot)$ and $\mathcal{U}(\cdot)$, is the polytope

$$\mathcal{X} = \{x \mid \mathcal{U}(x) \neq \emptyset\} = \{x \mid \exists\ u \in \mathcal{U}(x)\} = \mathrm{Proj}_X(\mathcal{Z}) \tag{B.5}$$

Let $p \geq l$ denote the number of rows of M, N and c. It is known that $V^0(\cdot)$ is continuous piecewise quadratic and continuous and $u^0(\cdot)$ is piecewise affine and continuous, being quadratic and affine, respectively, in the polytopes R_I, $I \subseteq \mathcal{I}_p$ that constitute a polytopic partition of $\mathcal{X}$. Each region is characterized by a set of active constraints I, i.e. for all $x \in R_I$:

$$M_I u^0(x) = N_I x + c_I \tag{B.6}$$

$$M_i u \leq N_i x + p_i \text{ for all } i \in I^c \tag{B.7}$$

$$-\nabla_u V(x, u^0(x)) \in PC_I(x) \tag{B.8}$$

where M_I, N_I and c_I denote the matrices with, rows M_i, N_i and c_i, respectively, $i \in I$, and $PC_I(x) \triangleq \{M_I'\lambda \mid \lambda \geq 0\}$ is the polar cone to the cone $F(x) \triangleq \{h \mid M_I h \leq 0\}$ of feasible directions at x; for each $I \subseteq \mathcal{I}_p$, I^c denotes the complement of I in $\mathcal{I}_p$. Thus $V^0(\cdot)$ is continuously differentiable (in fact analytic) in the interior of each region R_I, $I \subseteq \mathcal{I}_p$. We may assume, without loss of generality, that M_I has maximal rank. Our final assumption is:

A2: For any two adjacent regions R_{I_1} and R_{I_2} ($R_{I_1} \cap R_{I_2} \neq \emptyset$) either $I_1 \subset I_2$ or $I_1 \supset I_2$.

Assumption **A2** will often be satisfied, but there do exist counterexamples.

Theorem 8 *Suppose $V(\cdot)$ is continuously differentiable and that assumptions **A1** – **A2** are satisfied. Then $V^0(\cdot)$ is continuously differentiable in $\mathcal{X}$.*

Proof: It is known that $V^0(\cdot)$ is continuous piecewise quadratic and continuous and $u^0(\cdot)$ is piecewise affine and continuous, being quadratic and affine, respectively, in the polytopes R_I, $I \subseteq \mathcal{I}_p$ that constitute a polytopic partition of $\mathcal{X}$. Each region is characterized by a set of active constraints I, i.e. R_I is defined by the inequalities (B.6)-(B.8). Thus $V^0(\cdot)$ is continuously differentiable (in fact analytic) in the interior of each region R_I, $I \subseteq \mathcal{I}_p$. Consider the continuous differentiability of $V^0(\cdot)$ on the boundary between two regions R_{I_1} and R_{I_2} say where $I_1 \subseteq I_2$. For any $I \subseteq \mathcal{I}_p$ such that $R_I \neq \emptyset$, any $x \in R_I$,

$$u^0(x) = \tilde{\mathbf{u}}_I^0(x) + \bar{\mathbf{u}}_I^0(x) \tag{B.9}$$

where, for each index set I, $\tilde{\mathbf{u}}_I^0(x) \in \text{range}(M_I')= \mathcal{N}(M_I)^\perp$ (the row space of M_I) is that u of minimum norm satisfying $M_I u = N_I x + c_I$ and $\bar{\mathbf{u}}_I^0(x) \in \text{range}(M_I')^\perp = \mathcal{N}(M_I)$ ($\mathcal{N}(M_I)$ is the null space of M_I). Roughly speaking, $\tilde{\mathbf{u}}_I^0(x)$ satisfies the constraints, and $\bar{\mathbf{u}}_I^0(x)$ optimizes. It is easily shown that both $\tilde{\mathbf{u}}_I(\cdot)$ and $\bar{\mathbf{u}}_I^0(\cdot)$ are affine in x, satisfying, respectively

$$\tilde{\mathbf{u}}_I^0(x) = \tilde{K}_I x + \tilde{k}_I, \quad \bar{\mathbf{u}}_I^0(x) = \bar{K}_I x + \bar{k}_I \tag{B.10}$$

where $\tilde{k}_I$ and the columns of $\tilde{K}_I$ lie in range(M_I') and $\bar{k}_I$ and the columns of $\bar{K}_I$ lie in range$(M_I')^\perp$. In fact, $\tilde{K}_I$ and $\tilde{k}_I$ are given by

$$\tilde{K}_I = M_I^\dagger N_I, \quad \tilde{k}_I = M_I^\dagger c_I \tag{B.11}$$

where $M_I^\dagger$, the Moore-Penrose pseudo inverse of M_I, is given by

$$M_I^\dagger = (M_I' M_I)^{-1} M_I' \tag{B.12}$$

Since $V^0(\cdot)$ is continuously differentiable in each region R_I, consider the continuous differentiability of $V^0(\cdot)$ on the boundary between two regions, R_{I_1} and R_{I_2} say, where $I_1 \subseteq I_2$ Because, for $x \in R_I$, $u^0(x)$ minimizes (with respect to u) the continuously differentiable function $V(x,u)$ in the hyperplane $\{u \mid M_{I_1} u = N_{I_1} x + c_{I_1}\}$, we have

$$-\nabla_u V(x, u^0(x)) \in \{M_{I_1}' \lambda \mid \lambda \geq 0\} \subseteq \text{range}(M_{I_1}') \tag{B.13}$$

Hence

$$(\partial/\partial x) V_x^0(x) = (\partial/\partial x) V(x, u^0(x)) + (\partial/\partial u) V(x, u^0(x))(\partial/\partial x)\tilde{\mathbf{u}}_{I_1}^0(x) \tag{B.14}$$

since $(\partial/\partial u)V((x,u^0(x)))(\partial/\partial x)\bar{\mathbf{u}}_{I_1}^0(x) = 0$ (because of (B.13) and the fact that $(\partial/\partial x)\bar{\mathbf{u}}_{I_1}^0(x) = \bar{K}_{I_1}$) and the columns of $\bar{K}_{I_1}$ lie in range$(M_{I_1}')^\perp$. Suppose now $x \to x^* \in R_{I_1} \cap R_{I_2}$, $x \in R_{I_1}$. Then

$$(\partial/\partial x) V^0(x) \to (\partial/\partial x) V(x^*, u^0(x^*)) + (\partial/\partial u) V(x, u^0(x^*))\tilde{K}_{I_1} \tag{B.15}$$

where both $\nabla_u V(x, u^0(x^*))'$ and the columns of $\tilde{K}_{I_1} = (\partial/\partial x)\tilde{\mathbf{u}}_{I_1}^0(x^*)$ lie in range(M_{I_1}'). Next consider a $x \in R_{I_2}$ such that $x \to x^* \in R_{I_1} \cap R_{I_2}$. Arguing as above we deduce

$$(\partial/\partial x) V_x^0(x) \to (\partial/\partial x) V(x^*, u^0(x^*)) + (\partial/\partial u) V_u((x, u^0(x^*))\tilde{K}_{I_2} \tag{B.16}$$

where $\nabla_u V(x, u^0(x^*))$ lies in range(M'_{I_1}) (as above) but the columns of $\bar{K}_{I_2} = (\partial/\partial x)\tilde{\mathbf{u}}^0_{I_2}(x^*)$ lie in range(M'_{I_2}). We show below that $I_1 \subseteq I_2$ implies that $M_{I_1}\tilde{K}_{I_1} = M_{I_1}\tilde{K}_{I_2}$. Since $\nabla_u V((x, u^0(x^*))$ lies in range(M'_{I_1}), it follows that

$$(\partial/\partial u)V(x, u^0(x^*))\tilde{K}_{I_1} = (\partial/\partial u)V_u((x, u^0(x^*))\tilde{K}_{I_2} \tag{B.17}$$

Equations (B.15) - (B.17) establish the continuous differentiability of $V^0(\cdot)$ at $x^* \in R_{I_1} \cap R_{I_2}$.

We have now to show that $I_1 \subseteq I_2$ implies that $M_{I_1}\tilde{K}_{I_1} = M_{I_1}\tilde{K}_{I_2}$. Suppose

$$M_{I_2} = \begin{bmatrix} M_{I_1} \\ m \end{bmatrix}, \quad N_{I_2} = \begin{bmatrix} N_{I_1} \\ n \end{bmatrix},$$

Then, from (B.11)

$$M_{I_1}\tilde{K}_{I_2} = M_{I_1}M^{\dagger}_{I_2}N_{I_2}$$

But

$$M_{I_2}\tilde{K}_{I_2} = N_{I_2}$$

so that

$$\begin{bmatrix} M_{I_1} \\ m \end{bmatrix}\tilde{K}_{I_2} = \begin{bmatrix} N_{I_1} \\ n \end{bmatrix}$$

from which it follows that

$$M_{I_1}\tilde{K}_{I_2} = N_{I_1} = M_{I_1}\tilde{K}_{I_1}.$$

It follows from **A2** that $V^0(\cdot)$ is continuously differentiable in $\mathcal{X}$.

■

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