Optimization Over State Feedback Policies for Robust Control with Constraints *

Paul J. Goulart, Eric C. Kerrigan and Jan M. Maciejowski

Department of Engineering
University of Cambridge
Cambridge CB2 1PZ
United Kingdom
{pjg42,eck21,jmm}@eng.cam.ac.uk

March 15, 2005
Revised: August 1, 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems, which are subject to unknown but bounded state disturbances and mixed constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with memory of prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. This result implies that a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine state feedback policies, can be solved in a computationally efficient fashion using convex optimization methods without having to introduce any conservatism in the problem formulation. This equivalence result is used to design a robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and the constraints are satisfied for all time and for all allowable disturbance sequences. The cost that is chosen to be minimized in the associated finite horizon optimal control problem is a quadratic function in the disturbance-free state and input sequences. It is shown that the value of the receding horizon control law can be calculated at each sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic or given by a 1-norm or ∞-norm bound, or a second-order cone program (SOCP) if the disturbance set is ellipsoidal or given by a 2-norm bound.

*Technical Report CUED/F-INFENG/TR.494, Department of Engineering, University of Cambridge, UK.

1 Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim will be to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbance. This is a problem that has been studied for some time now in the optimal control literature [6–8, 58] and a number of different solutions are available, most of which draw on results from set invariance theory [10], ℓ_1 optimal control [16, 18, 52] or predictive control [3, 14, 42, 44, 45].

Predictive control is an optimal control technique that is widely employed in industry, mainly because of its strength in being able to handle constraints in a transparent fashion. In predictive control, a finite horizon optimal control problem is solved on-line at each time instant, given the latest plant measurements. The solution to this problem is then implemented in a variety of ways, depending on the application. Most often the solution is implemented in a receding horizon fashion, i.e. only the first part of the solution is applied and the same optimal control problem is re-solved at the next sample instant, but using an updated measurement. However, other types of implementation are also possible, such as variable horizon (the horizon length is also a decision variable), decreasing horizon (the horizon length decreases at each sample instant) and time-optimal (the shortest horizon length is chosen such that a target set can be reached at the end of the horizon).

It is now generally accepted that if disturbances are to be taken into account in the formulation of the optimal control problem, then the optimization has to be done over admissible state feedback policies, rather than open-loop input sequences, otherwise infeasibility and instability problems can occur [45]. However, optimization over arbitrary (nonlinear) feedback policies is particularly difficult if constraints have to be taken into account. Current proposals for achieving this using finite dimensional optimization, such as [51], are computationally intractable since the size of the optimization problem grows exponentially with the size of the problem data.

A number of results are also available that allow for the computation of the analytical solution to a large class of robust finite horizon optimal control problems with constraints. If all the relevant constraint sets are polyhedral and the cost function is suitably chosen, then the solution turns out to be a time-varying piecewise affine state feedback control policy [2, 11, 17, 31, 46, 50]. Unfortunately, the practicality of these results are currently limited to small problem sizes. This is because it is easy to find examples where the complexity of the solution to the optimal control problem grows exponentially with the size of the problem data.

As is often the case, there is very little one can do about the inherent complexity of the solution to a given control problem. The availability of a more efficient algorithm for computing the solution off-line or implementing it on-line will still not change the fact that at some point the off-line and/or on-line computation power will not be sufficient. A practically implementable solution somehow has to trade off optimality with computational complexity. Hence, a question that has received a lot of attention is how to formulate a suitable optimal control problem that allows for the efficient computation of a robustly stabilizing and admissible control policy.

An interesting recent proposal in the predictive control literature is to define a polytopic tube over the control horizon and compute control inputs only at the vertices of this tube [37]. Though this approach is computationally tractable and is equivalent to optimizing over a restricted class of piecewise affine state feedback control policies, it currently suffers from the drawback that it is not clear how to best choose the shape of the tube so as to minimize the conservativeness of the solution. Another possibility for reducing the on-line computational requirements is to compute a single dynamic linear controller off-line [13, 16], but this approach often results in controllers of very high dimension and can be very conservative in practice.

Hence, a popular approach in the predictive control literature is to compute one or more stabilizing linear state feedback control laws off-line and restrict the on-line computation to the selection of one of these control laws (if there are more than one), followed with the computation of a finite sequence of admissible perturbations to the selected control law [1,15,30,38,47]. Though this approach considerably reduces the computational complexity, it is not always obvious how to best choose the linear control laws off-line so as to minimize conservativeness.

An obvious improvement to this approach of "pre-stabilization" is to try to simultaneously compute the linear feedback control law and perturbation sequence on-line at each sample instant. However, the problem with this approach is that the predicted input and state sequences are often nonlinear functions of the sequence of state feedback gains. As a consequence, the set of feasible decision variables is non-convex, in general. Various proposals have been put forward for modifying this problem so that the set of feasible decision variables is convex [27, 36, 53], but generally this comes with an increase in the conservativeness of the solution. Hence, an interesting question is whether one can re-parameterize the optimal control problem, where the optimization is over the class of affine state feedback policies (linear feedback plus perturbation), and formulate an equivalent, but convex and tractable optimization problem. One of the contributions of this paper is to show that this is possible.

In order to show this, we exploit a recent result for solving a specific class of robust optimization problems with hard constraints, called adjustable robust counterpart (ARC) problems [5,23], where the optimization variables correspond to decisions that can be adjusted as soon as the actual value of the uncertainty becomes available. The authors of [5,23] proposed that instead of solving for an admissible nonlinear function of the uncertainty, one could aim to parameterize the solution as an affine function of the uncertainty. They proceeded to show that if the uncertainty set is a polyhedron and the constraints in the robust optimization problem are affine, then an affine function of the uncertainty can be found by solving a single, computationally tractable linear program (LP).

An equivalent parameterization was also proposed in the literature on predictive control with Gaussian state disturbances [55,56] and bounded state disturbances [40,41]. However, the question whether this alternative parameterization is more or less conservative than an affine state feedback parameterization, was not addressed in detail. It was also clear that further work needed to be undertaken in order to determine conditions on the resulting optimal control problem such that the solution can be used to synthesize robustly stabilizing predictive controllers with robust constraint satisfaction guarantees. Motivated by the encouraging results reported in [5,23,40,41,55,56], this paper presents a number of novel system-theoretic results relating to the use of this new parameterization.

In order for the results in this paper to be applicable to a large class of problems, the development of this paper starts with a general problem description, which is refined in each section. Section 2 discusses the class of systems that is to be considered throughout the paper and lists a number of standing assumptions. Section 3 describes the well-known affine *state* feedback parameterization and Section 4 describes the new affine *disturbance* feedback parameterization that was proposed in [5,23,40,41,55,56]. The key point made in Section 3 is that, in general, the set of admissible state feedback parameters is *non-convex.* In contrast, the main result in Section 4 states that the set of admissible disturbance feedback parameters is *convex.* Under suitable assumptions on the disturbance, it is then shown that an admissible disturbance feedback policy can be found by solving a single and tractable convex optimization problem.

The main contributions of this paper are contained in Sections 5–7. In Section 5 it is shown that the affine state feedback parameterization of Section 3 is *equivalent* to the affine disturbance feedback parameterization of Section 4. This has important system-theoretical consequences, which are explored in detail in Sections 6 and 7. Section 6 is mainly concerned with results that guarantee robust constraint satisfaction for all time. Section 7 formulates a suitable robust finite horizon optimal control problem and presents results that guarantee robust stability of the closed-loop system for the case when the solution to the optimal control problem is implemented in a receding horizon fashion. The paper concludes in Section 8 and suggests some topics for further research.

Some of the results in this paper have been published by the authors in the conference papers [21,30,32]. For a detailed discussion on the numerical implementation of the results in this paper, the reader is referred to the conference paper [20] and technical report [22].

Notation and definitions: For matrices A and B, $A \otimes B$ is the Kronecker product of A and B, $A^{\dagger}$ is the one-sided or pseudo-inverse of A, and $A \leq B$ denotes element-wise inequality and $\text{abs}(A)$ is the element-wise absolute value of A. A matrix, not necessarily square, is referred to as *(strictly) lower triangular* if the (i,j) entry is zero for all $i < j$ $(i \leq j)$. A block partitioned matrix is referred to as *(strictly) block lower triangular* if the (i,j) block is zero when $i < j$ $(i \leq j)$; note that a *block* lower triangular matrix is not necessarily lower triangular. $\mathbf{1}$ is a column vector of ones. For vectors x and y, $\text{vec}(x,y) := [x^T \; y^T]^T$, and $\|x\|_Q^2 := x^T Q x$. $\mathbb{Z}_{[k,l]}$ represents the set of integers $\{k, k+1, \ldots, l\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$-function if, in addition, $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if for all $k \geq 0$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function and for each $s \geq 0$, $\beta(s, \cdot)$ is decreasing with $\beta(s,k) \to 0$ as $k \to \infty$.

2 Standing Assumptions

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance[1]. The current and future values of the disturbance are unknown and may change unpredictably from one time instant to the next, but are contained in a convex and compact (closed and bounded) set W, which contains the origin. The actual values of the state, input and disturbance at time instant k are denoted by $x(k)$, $u(k)$ and $w(k)$, respectively; where it is clear from the context, x, u and w will be used to denote the current value of the state, input and disturbance (note that since the system is time-invariant, the current time can always be taken as zero). It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. We also assume that a linear state feedback gain matrix $K \in \mathbb{R}^{m \times n}$ is given, such that $A + BK$ is strictly stable.

The system is subject to mixed constraints on the state and input:

$$\mathcal{Z} := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{2}$$

where the matrices $C \in \mathbb{R}^{s \times n}$, $D \in \mathbb{R}^{s \times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints that define $\mathcal{Z}$. It is assumed that $\mathcal{Z}$ is bounded and contains the origin in its interior. A primary design goal is to guarantee that the state and input of the closed-loop system remain in $\mathcal{Z}$ for all time and for all allowable disturbance sequences.

In addition to $\mathcal{Z}$, a target/terminal constraint set X_f is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{3}$$

where the matrix $Y \in \mathbb{R}^{r \times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f. It is assumed that X_f is bounded and contains the origin in its interior. As will be seen in Sections 6 and 7, the set X_f can be used as a target set in time-optimal control or as a terminal constraint in a receding horizon controller with guaranteed invariance and stability properties.

Remark 1. Many of the results in this paper remain valid if the assumption that $\mathcal{Z}$ and X_f are polytopes is relaxed to $\mathcal{Z}$ and X_f being convex; the current assumptions serve to simplify the presentation in Section 4.

[1] This assumption on the disturbance is without loss of generality; the results in this paper are easily generalized to the case where $x^+ = Ax + Bu + Ed$ (see Section 4.2).

Before proceeding, we define some additional notation. In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{nN}$, respectively, as

$$\mathbf{x} := \operatorname{vec}(x_0, \ldots, x_{N-1}, x_N), \tag{4a}$$

$$\mathbf{u} := \operatorname{vec}(u_0, \ldots, u_{N-1}), \tag{4b}$$

$$\mathbf{w} := \operatorname{vec}(w_0, \ldots, w_{N-1}), \tag{4c}$$

where $x_0 = x$ denotes the current measured value of the state and $x_{i+1} := Ax_i + Bu_i + w_i$, $i = 0, \ldots, N-1$ denote the prediction of the state after i time instants into the future. Finally, let the set $\mathcal{W} := W^N := W \times \cdots \times W$, so that $\mathbf{w} \in \mathcal{W}$.

3 Affine State Feedback Parameterization

One natural approach to controlling the system in (1), while ensuring the satisfaction of the constraints, is to search over the set of time-varying affine state feedback control policies with memory of prior states:

$$u_i = \sum_{j=0}^{i} L_{i,j} x_j + g_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{5}$$

where each $L_{i,j} \in \mathbb{R}^{m \times n}$ and $g_i \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} := \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \tag{6a}$$

and

$$\mathbf{g} := \operatorname{vec}(g_0, \ldots, g_{N-1}), \tag{6b}$$

so that the control input sequence can be written as

$$\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}. \tag{7}$$

For a given initial state x, we say that the pair $(\mathbf{L}, \mathbf{g})$ is admissible if the control policy (5) guarantees that for all allowable disturbance sequences of length N, the constraints (2) are satisfied over the horizon $i = 0, \ldots, N-1$ and that the state is in the target set (3) at the end of the horizon. More precisely, the set of admissible $(\mathbf{L}, \mathbf{g})$ is defined as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \;\middle|\; \begin{array}{r} (\mathbf{L}, \mathbf{g}) \text{ satisfies (6)}, \; x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i} L_{i,j} x_j + g_i \\ (x_i, u_i) \in \mathcal{Z}, \; x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]}, \; \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{8}$$

The set of initial states x for which an admissible control policy of the form (5) exists is defined as

$$X_N^{sf} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{sf}(x) \neq \emptyset \right\}. \tag{9}$$

It is critical to note that it may not be possible to select a single $(\mathbf{L}, \mathbf{g})$ such that it is admissible for all $x \in X_N^{sf}$. Indeed, it is easy to find examples where there exists a pair $(x, \tilde{x}) \in X_N^{sf} \times X_N^{sf}$ such that $\Pi_N^{sf}(x) \bigcap \Pi_N^{sf}(\tilde{x}) = \emptyset$. For problems of non-trivial size, it is therefore necessary to calculate an admissible pair $(\mathbf{L}, \mathbf{g})$ on-line, given a measurement of the current state x, rather than fixing $(\mathbf{L}, \mathbf{g})$ off-line. Once an admissible control policy is computed for the current state, there are many ways in which it can be applied to the system; time-varying, time-optimal and receding-horizon implementations are the most common (these will be considered in more detail in Sections 6 and 7). We emphasize that, due to the dependence of (8) on the current state x, the implemented control policy will, in general, be a *nonlinear* function in x, even though it may have been defined in terms of the class of affine state feedback policies (5).

Remark 2. Note that the state feedback policy (5) subsumes the well-known class of "pre-stabilizing" control policies [1, 15, 30, 38, 47], in which the control policy takes the form $u_i = Kx_i + c_i$, where K is computed off-line and on-line computation is limited to finding an admissible offset sequence $\{c_i\}_{i=0}^{N-1}$.

Finding an admissible pair $(\mathbf{L}, \mathbf{g})$, given the current state x, has been believed to be a very difficult problem due to the following property:

Proposition 1 (Non-convexity). *For a given state $x \in X_N^{sf}$, the set of admissible affine state feedback control parameters $\Pi_N^{sf}(x)$ is non-convex, in general.*

This is easily shown by the following example:

Example 1. Consider the SISO system

$$x^+ = x + u + w \tag{10}$$

with initial state $x_0 = 0$, input constraint $|u| \leq 3$, bounded disturbances $|w| \leq 1$ and a planning horizon of $N = 3$. Consider a control policy of the form (5) with $\mathbf{g} = 0$ and $L_{2,1} = 0$, so that $u_0 = 0$ and

$$u_1 = L_{1,1} w_0 \tag{11}$$

$$u_2 = \left[L_{2,2}(1 + L_{1,1}) \right] w_0 + L_{2,2} w_1 \tag{12}$$

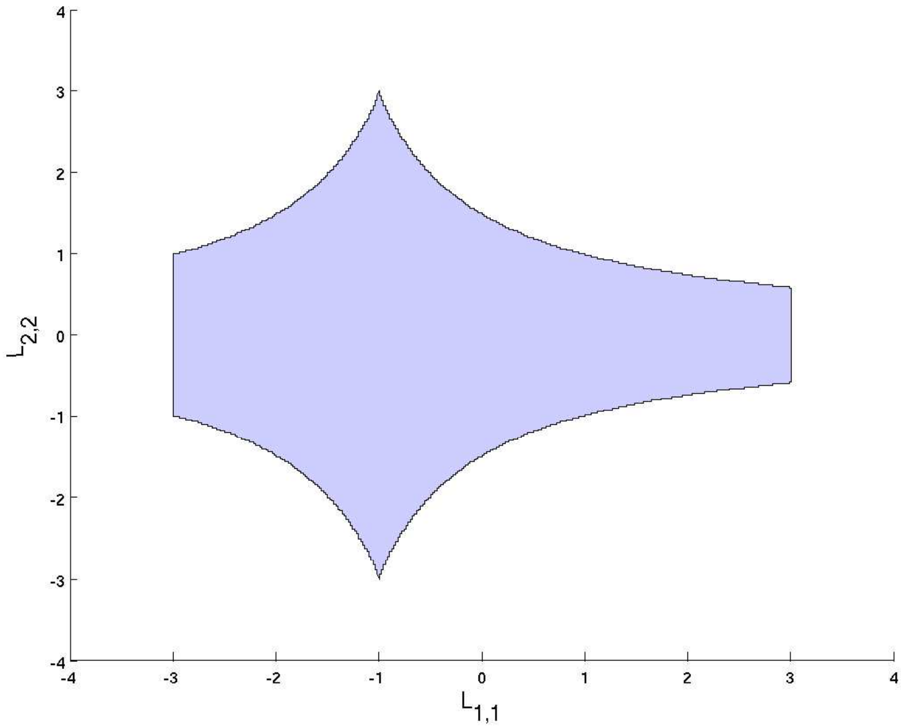

Figure 1: Non-Convexity of $\Pi_N^{sf}(0)$ in Example 1

In order to satisfy the input constraints for all allowable disturbance sequences, the controls u_i must satisfy

$$|u_i| \leq 3, \ i = 1, 2, \ \forall \mathbf{w} \in \mathcal{W} \tag{13}$$

or, equivalently,

$$\max_{\mathbf{w} \in \mathcal{W}} |u_i| \leq 3, \ i = 1, 2. \tag{14}$$

Since the constraints on the components of $\mathbf{w}$ are independent, it is easy to show that the input constraints are satisfied for all $\mathbf{w} \in \mathcal{W}$ if and only if

$$|L_{1,1}| \leq 3 \tag{15}$$

$$|L_{2,2}(1 + L_{1,1})| + |L_{2,2}| \leq 3. \tag{16}$$

It is straightforward to verify that the set of gains $\mathbf{L}$, which satisfy these constraints, is non-convex; the set of admissible values for $(L_{1,1}, L_{2,2})$ is shown in Figure 1.

Remark 3. It is surprising to note that, even though the set of admissible control parameters $\Pi_N^{sf}(x)$ may be non-convex, the set of states X_N^{sf}, for which at least one admissible control parameter exists, is *always* convex. We defer the proof of this until Section 5.

Despite the fact that the set of admissible parameters $\Pi_N^{sf}(x)$ may be non-convex, we will proceed to show that one can actually find an admissible $(\mathbf{L}, \mathbf{g})$ by solving a single, *tractable* and *convex* programming problem.

4 Affine Disturbance Feedback Parameterization

An alternative to (5) is to parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{17}$$

where each $M_{i,j} \in \mathbb{R}^{m \times n}$ and $v_i \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance sequence is easily calculated as the difference between the predicted and actual states at each step, i.e.

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}. \tag{18}$$

The above parameterization appears to have originally been suggested some time ago within the context of stochastic programs with recourse [19]. More recently, it has been revisited as as a means for finding solutions to a class of robust optimization problems, called affinely adjustable robust counterpart (AARC) problems [5, 23], and robust model predictive control problems [40, 41, 55, 56].

Though the two parameterizations in (5) and (17) look very similar (in fact, it will be shown in Section 5 that they are *equivalent*), they have different interpretations. Furthermore, as will be seen in subsequent sections, in some cases it is easier to prove certain system-theoretic results using one parameterization, rather than the other. In this section, we will discuss one of the main benefits of adopting the parameterization in (17), namely that an admissible affine disturbance feedback control law can be found by solving a convex and tractable optimization problem.

For notational convenience, we define the vector $\mathbf{v} \in \mathbb{R}^{mN}$ and the strictly block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix} \tag{19a}$$

and

$$\mathbf{v} := \mathrm{vec}(v_0, \ldots, v_{N-1}), \tag{19b}$$

so that the control input sequence can be written as

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}. \tag{20}$$

For a given initial state x, we say that the pair $(\mathbf{M}, \mathbf{v})$ is admissible if the control policy (17) guarantees that for all allowable disturbance sequences of length N, the constraints (2) are satisfied over the horizon $i = 0, \ldots, N-1$ and that the state is in the target set (3) at the end of the horizon. More precisely, the set of admissible $(\mathbf{M}, \mathbf{v})$ is defined as

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (19)},\ x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \\ (x_i, u_i) \in \mathcal{Z},\ x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]},\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\} \tag{21}$$

The set of initial states x for which an admissible control policy of the form (17) exists is defined as

$$X_N^{df} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{df}(x) \neq \emptyset \right\}. \tag{22}$$

Before proceeding, it is important to note that one can easily find matrices $F \in \mathbb{R}^{t \times mN}$, $G \in \mathbb{R}^{t \times nN}$, $H \in \mathbb{R}^{t \times n}$ and a vector $c \in \mathbb{R}^t$, where $t := sN + r$ (for completeness, the matrices and vectors are given in the Appendix), such that one can rewrite the expression for $\Pi_N^{df}(x)$ more compactly as

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (19)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Hx,\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{23}$$

4.1 Convexity of $\Pi_N^{df}(x)$

The main advantage of the disturbance feedback parameterization in (17) over the state feedback parameterization in (5) is formalized in the following statement:

Proposition 2 (Convexity). *For a given state $x \in X_N^{df}$, the set of admissible affine disturbance feedback parameters $\Pi_N^{df}(x)$ is convex and closed. Furthermore, the set of states X_N^{df}, for which at least one admissible affine disturbance feedback policy exists, is convex.*

Proof. Consider the set

$$\mathcal{C}_N := \left\{ (\mathbf{M}, \mathbf{v}, x) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (19)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Hx \\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{24}$$

It immediately follows that

$$\mathcal{C}_N = \bigcap_{\mathbf{w} \in \mathcal{W}} \left\{ (\mathbf{M}, \mathbf{v}, x) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (19)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Hx \end{array} \right\}.$$

$\mathcal{C}_N$ is closed and convex, since it is the intersection of an arbitrary collection of closed and convex sets. The set X_N^{df} is convex since it is a projection of $\mathcal{C}_N$ onto a suitably-defined subspace. Since the set $\Pi_N^{df}(x)$ in (23) can similarly be written as an intersection of closed and convex sets, it is also closed and convex. $\square$

The above result is of fundamental importance. If $\mathcal{W}$ is convex and compact, then it is conceptually possible to compute a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ in a computationally tractable way, given the current state x.

Recall that $a^T\mathbf{w} \leq e$ for all $\mathbf{w} \in \mathcal{W}$ if and only if $\sup\left\{a^T\mathbf{w} \mid \mathbf{w} \in \mathcal{W}\right\} \leq e$, where a is a vector of appropriate length, e is a scalar and $\sup\left\{a^T\mathbf{w} \mid \mathbf{w} \in \mathcal{W}\right\}$ is the value of the *support function* of $\mathcal{W}$ evaluated at a [35]; note that the supremum is attained due to $\mathcal{W}$ being compact. Hence, one can eliminate the universal quantifier in (23) to obtain the equivalent expression

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (19)} \\ F\mathbf{v} + \operatorname{vec}\max\limits_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)\mathbf{w} \leq c + Hx \end{array} \right\}, \tag{25}$$

where $\operatorname{vec}\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M}+G)\mathbf{w}$ denotes row-wise maximization, i.e. if $(F\mathbf{M}+G)_i$ denotes the i^{th} row of the matrix $F\mathbf{M} + G$, then

$$\operatorname{vec}\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)\mathbf{w} := \operatorname{vec}\left(\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)_1\mathbf{w}, \ldots, \max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)_t\mathbf{w}\right). \tag{26}$$

Computing an $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ is done by formulating the dual optimization problem of each maximization problem $\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)_i\mathbf{w}$, $i = 1, \ldots, t$, introducing some slack variables and solving a single, suitably-defined convex programming problem. For example, if $\mathcal{W}$ is a closed and bounded polyhedron given by a finite set of affine inequalities, then one can compute an admissible pair $(\mathbf{M}, \mathbf{v})$ by solving a single linear program (LP), of which the number of decision variables and number of constraints are polynomial functions of the size of the data; the reader is referred to [5, Thm. 3.2] and [23, Thm. 4.2] for details. Clearly, other suitable convex optimization problems can be derived for alternative descriptions of the disturbance set $\mathcal{W}$, but it is beyond the scope of this paper to present all the details. However, for completeness we will proceed to give some specific results for commonly-encountered descriptions of the set of allowable disturbances.

Remark 4. Note that the proof of Proposition 2 does not require $\mathcal{W}$ to be convex. However, convexity of $\mathcal{W}$ is important for the efficient computation of an admissible pair $(\mathbf{M}, \mathbf{v})$.

4.2 Admissible Policies for Norm-Bounded Disturbances

It is particularly easy to find a pair $(\mathbf{M}, \mathbf{v})$ in (25) for the special case where the disturbance set W (or, more generally, $\mathcal{W}$) represents the affine map of a set of norm-bounded signals:

$$W = \{ w \in \mathbb{R}^n \mid w = Ed + f, \; \|d\|_p \leq 1 \}, \tag{27}$$

where $E \in \mathbb{R}^{n \times l}$ and $f \in \mathbb{R}^n$. If this is the case, then an analytical expression for the solution to the row-wise maximization in (25) is easily found. From the definition of the dual norm [24], it immediately follows that if W is given as in (27), then

$$\max_{w \in W} a^T w = \|E^T a\|_q + a^T f, \tag{28}$$

for any vector $a \in \mathbb{R}^n$, where $1/p + 1/q = 1$.

As an example, we give a detailed exposition for the ∞-norm case with $f = 0$ (as in (28), the extension to the case when $f \neq 0$ is trivial). Suppose that the disturbance set W is the linear map of a hypercube, so that the disturbances satisfy

$$W = \{ Ed \mid \|d\|_\infty \leq 1 \}, \tag{29}$$

then the row-wise maximization in (25) can be simplified using (28) (with $q = 1$) to yield

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) &\text{ satisfies (19)} \\ F\mathbf{v} + \operatorname{abs}(F\mathbf{M}J + GJ)\mathbf{1} &\leq c + Hx \end{aligned} \right\}, \tag{30}$$

where $J := I \otimes E \in \mathbb{R}^{Nn \times Nl}$; note that $\operatorname{abs}(F\mathbf{M}J + GJ)\mathbf{1}$ is a vector formed from the 1-norms of the rows of the matrix $F\mathbf{M}J + GJ$. The above expression can be written in terms of a set of purely affine constraints by following a standard procedure and introducing some slack variables to get

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{aligned} (\mathbf{M}, \mathbf{v}) \text{ satisfies (19) and } \exists \mathbf{\Lambda} \in \mathbb{R}^{t \times nN} &\text{ s.t.} \\ F\mathbf{v} + \mathbf{\Lambda}\mathbf{1} &\leq c + Hx \\ -\mathbf{\Lambda} \leq (F\mathbf{M}J + GJ) &\leq \mathbf{\Lambda} \end{aligned} \right\}. \tag{31}$$

An admissible pair $(\mathbf{M}, \mathbf{v})$ can then be found by solving a single (Phase I) LP with at most $N^2(mn + sl) + N(m + rl) + 1$ variables and $2N^2 sl + N(s + 2rl) + r$ inequality constraints, many of which may be eliminated by accounting for the block lower triangular structure of $\mathbf{M}$, F and G. The translation of the problem into a form suitable to be passed to a standard LP solver is achieved by applying some basic Kronecker product identities.

A similar process leads, in the case of 1-norm bounded disturbances to another tractable LP. In the case of 2-norm bounds (e.g. if W is the affine map of a Euclidean ball or an ellipsoid), this leads to a second order cone program (SOCP) with a polynomial number of decision variables and constraints.

5 Equivalence between State and Disturbance Feedback Parameterizations

One question that has not yet been answered in the literature, is whether the disturbance feedback parameterization (17) is more conservative or less conservative than the state feedback parameterization (5). We now show that these parameterizations are actually equivalent:

Theorem 1 (Equivalence). *The set of admissible states* $X_N^{df} = X_N^{sf}$*. Additionally, given any* $x \in X_N^{sf}$*, for any admissible* $(\mathbf{L}, \mathbf{g})$ *an admissible* $(\mathbf{M}, \mathbf{v})$ *can be found which yields the same input and state sequence for all allowable disturbance sequences, and vice-versa.*

Proof. $X_N^{sf} \subseteq X_N^{df}$: By definition, for a given $x \in X_N^{sf}$, there exists a pair $(\mathbf{L}, \mathbf{g})$ that satisfies the constraints in (8). For a given disturbance sequence $\mathbf{w} \in \mathcal{W}$, the inputs and states of the system can be written as :

$$\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g} \tag{32}$$

$$\mathbf{x} = \mathbf{B}(\mathbf{L}\mathbf{x} + \mathbf{g}) + \mathbf{E}\mathbf{w} + \mathbf{A}x \tag{33}$$

$$= (I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{g} + \mathbf{E}\mathbf{w} + \mathbf{A}x) \tag{34}$$

where $x_0 = x$, $\mathbf{x} := \mathrm{vec}(x_0, \ldots, x_N)$, and the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$ (for completeness, these are given in the Appendix) are defined so that one can write

$$\mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}. \tag{35}$$

The matrix $I - \mathbf{B}\mathbf{L}$ is always non-singular, since $\mathbf{B}\mathbf{L}$ is strictly block lower triangular. The control sequence can then be rewritten as an affine function of the disturbance sequence $\mathbf{w}$:

$$\mathbf{u} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{g} + \mathbf{A}x) + \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w} + \mathbf{g}, \tag{36}$$

and an admissible $(\mathbf{M}, \mathbf{v})$ constructed by choosing

$$\mathbf{M} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}\mathbf{E} \tag{37a}$$

$$\mathbf{v} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{g} + \mathbf{A}x) + \mathbf{g}. \tag{37b}$$

This choice of $(\mathbf{M}, \mathbf{v})$ gives exactly the same input sequence as the pair $(\mathbf{L}, \mathbf{g})$, so the state and input constraints in (21) are satisfied. The constraint (19) that $\mathbf{M}$ be strictly block lower triangular is satisfied because $\mathbf{M}$ is chosen in (37) as a product of the block lower triangular matrices $(I - \mathbf{B}\mathbf{L})^{-1}$ and $\mathbf{L}$ and the strictly block lower triangular matrix $\mathbf{E}$. Therefore, $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ and thus $x \in X_N^{sf} \Rightarrow x \in X_N^{df}$.

$X_N^{df} \subseteq X_N^{sf}$: By definition, for a given $x \in X_N^{df}$, there exists a pair $(\mathbf{M}, \mathbf{v})$ that satisfies the constraints in (21). For a given disturbance sequence $\mathbf{w} \in \mathcal{W}$, the inputs and states of the system can be written as :

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v} \tag{38}$$

$$\mathbf{x} = \mathbf{B}(\mathbf{M}\mathbf{w} + \mathbf{v}) + \mathbf{E}\mathbf{w} + \mathbf{A}x \tag{39}$$

Recall that since full state feedback is assumed, the disturbances can be determined exactly from

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{40}$$

which can be written in matrix form as

$$\mathbf{w} = \begin{bmatrix} 0 & I & 0 & \cdots & \cdots & 0 \\ 0 & -A & I & 0 & \ddots & \vdots \\ 0 & 0 & -A & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -A & I \end{bmatrix} \mathbf{x} - \begin{bmatrix} I \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} Ax + (I \otimes B)\mathbf{u}, \tag{41}$$

or more compactly as

$$\mathbf{w} = \mathbf{E}^\dagger \mathbf{x} - \mathcal{I}Ax + \mathbf{E}^\dagger \mathbf{B}\mathbf{u}, \tag{42}$$

where $\mathcal{I} := [I \quad 0 \quad \cdots \quad 0]^T$. It is easy to verify that the matrices $\mathbf{E}^\dagger$ and $\mathcal{I}^T$ are left inverses of $\mathbf{E}$ and $\mathbf{A}$ respectively, so that $\mathbf{E}^\dagger\mathbf{E} = I$ and $\mathcal{I}^T\mathbf{A} = I$.

The input sequence can then be rewritten as

$$\begin{aligned} \mathbf{u} &= \mathbf{M}(\mathbf{E}^\dagger \mathbf{x} - \mathcal{I}Ax + \mathbf{E}^\dagger \mathbf{B}\mathbf{u}) + \mathbf{v} && (43) \\ &= (I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B})^{-1}(\mathbf{M}\mathbf{E}^\dagger\mathbf{x} - \mathbf{M}\mathcal{I}Ax + \mathbf{v}). && (44) \end{aligned}$$

The matrix $I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B}$ is non-singular because the product $\mathbf{M}\mathbf{E}^\dagger\mathbf{B} = \mathbf{M}(I \otimes B)$ is strictly block lower triangular. An admissible $(\mathbf{L}, \mathbf{g})$ can then be constructed by choosing

$$\begin{aligned} \mathbf{L} &= (I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B})^{-1}\mathbf{M}\mathbf{E}^\dagger && (45a) \\ \mathbf{g} &= (I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B})^{-1}(\mathbf{v} - \mathbf{M}\mathcal{I}Ax). && (45b) \end{aligned}$$

This choice of $(\mathbf{L}, \mathbf{g})$ gives exactly the same input sequence as the pair $(\mathbf{M}, \mathbf{v})$, so the state and input constraints in (8) are satisfied. The constraint that $\mathbf{L}$ be block lower triangular is satisfied because it is the product of block lower triangular matrices. Therefore, $(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)$ and thus $x \in X_N^{df} \Rightarrow x \in X_N^{sf}$. □

In light of the discussion in Sections 3 and 4, it is crucial to note an important consequence of Theorem 1. The problem of finding an admissible affine state feedback control policy, which for a long time has been thought to be extremely difficult to solve because it gives rise to a non-convex optimization problem, can actually be solved efficiently using convex programming. This is done by re-parameterizing the problem and interpreting it as finding an affine disturbance feedback policy; given an admissible $(\mathbf{M}, \mathbf{v})$, an admissible $(\mathbf{L}, \mathbf{g})$ is given by (45).

It is important to note that Theorem 1 may not hold under certain structural constraints on $(\mathbf{L}, \mathbf{g})$ and/or $(\mathbf{M}, \mathbf{v})$. In some cases it is possible to prove inclusion in one direction, but it is not always obvious whether inclusion holds in the opposite direction. This is because the nonlinear transformations in (37) and (45) only allow a limited number of structural constraints to be preserved. For example, in [31] the case when $L_{i,j} = 0$ for all $i \neq j$ is considered (this corresponds to affine state feedback with no memory) and in [41, Sect. 7.5] the constraint that $L_{i,i} = L_{j,j}$ for all $i \neq j$ is added; in both these cases it is possible to show that affine disturbance feedback of the form (17) subsumes affine state feedback, but a proof in the opposite direction is lacking. Problems are also encountered if one enforces similar linear constraints on $\mathbf{M}$; in some cases it is easy to find counter-examples to an equivalence claim.

The availability of the equivalence statement in Theorem 1 allows one to make a number of strong system-theoretic statements about system (1) in closed-loop with a controller that has been derived from an admissible affine disturbance feedback control policy. As will be seen in the sequel, the proofs of some statements are relatively straightforward if one is allowed to move freely between state and disturbance feedback parameterizations. An interesting topic for further research would be to determine which linear constraints on the structure of $(\mathbf{L}, \mathbf{g})$ or $(\mathbf{M}, \mathbf{v})$ can be added, while still being able to prove equivalence as in Theorem 1 and derive all of the system-theoretic results given in Sections 6 and 7.

We conclude this section by comparing Theorem 1 with Proposition 2. This leads immediately to the following result that, in the light of Proposition 1, is rather surprising:

Corollary 1 (Convexity of X_N^{sf}). *The set of states X_N^{sf}, for which an admissible affine state feedback policy of the form* (5) *exists, is a convex set.*

6 Geometric and Invariance Properties

It is well-known that the set of states for which an admissible open-loop input sequence exists (i.e. when $\mathbf{L} = 0$ and $\mathbf{M} = 0$) may collapse to the empty set if the horizon is sufficiently large [51, Sect. F]. Furthermore, unless additional assumptions are made, the use of a finite planning horizon does not allow one to guarantee that if one can find an admissible feedback policy at the initial time instant, then one can find an admissible feedback policy at all future time instants. Likewise, putting arbitrary structural constraints on the feedback components $\mathbf{L}$ or $\mathbf{M}$ may also lead to infeasibility or instability of the closed-loop system.

The aim of this section is to provide conditions under which constraint satisfaction problems will not occur (stability will be considered in Section 7). Once an admissible affine feedback policy has been computed, there are many ways in which the control policy may be applied to the system. In this section, we will restrict ourselves to time-varying, receding-horizon and time-optimal implementations.

Before proceeding, we introduce the following standard assumption (c.f. [45]):

Assumption 1 (Terminal constraint). *The state feedback gain matrix K and terminal constraint X_f have been chosen such that:*

- *X_f is contained inside the set of states for which the constraint* (2) *is* *satisfied under the control $u = Kx$, i.e.*

$$X_f \subseteq \{x \mid (x, Kx) \in \mathcal{Z}\} = \{x \mid (C + DK)x \leq b\}. \tag{46}$$

- *X_f is robust positively invariant for the closed-loop system $x^+ = (A + BK)x + w$, i.e.*

$$(A + BK)x + w \in X_f, \quad \forall x \in X_f,\ \forall w \in W. \tag{47}$$

Under some additional, mild technical assumptions, it is easy to compute a K and a polytopic X_f that satisfies Assumption 1 if W is a polytope, an ellipsoid or the affine map of a p-norm ball. The reader is referred to [9,10,35,38,49] and the references therein for details.

6.1 Monotonicity of X_N^{sf} and X_N^{df}

We are now in a position to give a sufficient condition under which one can guarantee that X_N^{sf} (equivalently, X_N^{df}) is non-empty and the size of X_N^{sf} is non-decreasing (with respect to set inclusion) with horizon length N:

Proposition 3 (Size of X_N^{sf}). *If Assumption 1 holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^{sf} \subseteq \cdots \subseteq X_{N-1}^{sf} \subseteq X_N^{sf} \subseteq X_{N+1}^{sf} \subseteq \cdots, \tag{48}$$

where each X_i^{sf} is defined as in (9) with $N = i$.

Proof. The proof is by induction. Let $x \in X_N^{sf}$, $(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)$ and $\mathbf{w} \in \mathcal{W}$. It is simple to construct a pair $(\bar{\mathbf{L}}, \bar{\mathbf{g}}) \in X_{N+1}^{sf}$ by selecting $\bar{\mathbf{g}} \in \mathbb{R}^{m(N+1)}$ and $\bar{\mathbf{L}} \in \mathbb{R}^{m(N+1)\times n(N+2)}$ such that the final stage input will be $u_N = Kx_N$; such a pair is given by $\bar{\mathbf{g}} := \mathrm{vec}(\mathbf{g}, 0)$ and

$$\bar{\mathbf{L}} := \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 & 0 \\ 0 & \cdots & 0 & K & 0 \end{bmatrix}. \tag{49}$$

From the definition of $\Pi_N^{sf}(x)$, it follows that $x_N \in X_f$. Note also that since $X_f \subseteq X_K$, it follows that $(x_N, u_N) \in \mathcal{Z}$. Since X_f is robust positively invariant for the closed-loop system $x^+ = (A+BK)x+w$, it follows that $x_{N+1} = Ax_N + Bu_N + w_N \in X_f$, $\forall w_N \in W$. It then follows from the definition of $\Pi_{N+1}^{sf}(x)$ that $(\bar{\mathbf{L}}, \bar{\mathbf{g}}) \in \Pi_{N+1}^{sf}(x)$, hence $x \in X_{N+1}^{sf}$. The proof is completed by verifying, in a similar manner, that $X_f \subseteq X_1^{sf} \subseteq X_2^{sf}$. □

Remark 5. For many examples, some of the inclusions in (48) are strict, rather than satisfied with equality. Note also that if $X_N^{sf} = X_{N+1}^{sf}$, for some N, then $X_i^{sf} = X_N^{sf}$ for all $i > N$.

Remark 6. Note that we cannot use the same method of proof as in [46, Thm. 2], since we are not considering the same problem of finding a sequence of time-varying nonlinear state feedback control laws. By restricting the control laws to be affine, the arguments in [46, Thm. 2] cannot be applied (a similar problem is encountered in open-loop robust MPC, where $\mathbf{L} = 0$ [45, 51]). Instead, the proof follows a procedure similar to the one often employed to prove recursive feasibility and monotonicity of the value function in predictive control [45]. However, in contrast to [45], we are interested in proving monotonicity of the set of admissible states X_N^{sf}. Hence, rather than using a shifted version of the input sequence, we *append* the input sequence with the terminal control law $u = Kx$.

The next result follows immediately:

Corollary 2 (Size of X_N^{df}). *If Assumption 1 holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^{df} \subseteq \cdots \subseteq X_{N-1}^{df} \subseteq X_N^{df} \subseteq X_{N+1}^{df} \subseteq \cdots, \tag{50}$$

where each X_i^{df} is defined as in (22) *with $N = i$.*

Remark 7. Corollary 2 should be compared with the equivalent result in [32, Thm. 2]. The proof given here is more transparent, due to the application of Theorem 1 and Proposition 3.

Remark 8. On examination of the proof of Proposition 3, it is obvious that (48) still holds under some constraints on the structure of $\mathbf{L}$, e.g. if $L_{i,j} = 0$ for some $i \neq j$. However, it is important to note that the method of proof for Proposition 3 is only valid for a limited number of structural constraints on $\mathbf{L}$. If $\mathbf{L}$ is required to be block-Toeplitz and banded (for example, if $L_{i,j} = 0$ and $L_{i,i} = L_{j,j}$ for all $i \neq j$), then the method of proof given for Proposition 3 is clearly not applicable and the result may therefore not hold, in general. Similarly, it is important to note that (50) may not hold if we enforce any additional, linear constraints on the structure of $\mathbf{M}$, such as requiring that $\mathbf{M}$ be banded and/or block-Toeplitz.

6.2 Time-varying Control Laws

In this section we consider what happens if one were to implement an admissible affine disturbance feedback policy in a time-varying fashion. In other words, given any $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x(0))$ and the stabilizing state feedback gain $K \in \mathbb{R}^{m \times n}$, consider the following *time-varying* affine disturbance feedback policy:

$$u(k) = \begin{cases} v_k + \sum_{j=0}^{k-1} M_{i,j} w(j) & \text{if } k \in \mathbb{Z}_{[0,N-1]} \\ Kx(k) & \text{if } k \in \mathbb{Z}_{[N,\infty)} \end{cases} \tag{51}$$

Recall that the realized disturbance sequence $w(\cdot)$ can of course be recovered using the relation $w(k) = x(k+1) - Ax(k) - Bu(k)$. Obviously, Theorem 1 implies that we could also have defined an equivalent, time-varying affine state feedback policy. However, because of Propositions 1 and 2, it is more practical to think in terms of disturbance feedback policies when it comes to computation and implementation.

The next result follows immediately, by employing the state feedback parameterization (5) and recalling Theorem 1:

Proposition 4 (Time-varying control). *Let Assumption 1 hold, the initial state* $x(0) \in X_N^{df}$ *and* $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x(0))$*. For all allowable infinite disturbance sequences, the state of system* (1)*, in closed-loop with the feedback policy* (51)*, enters* X_f *in* N *steps or less and is in* X_f *for all* $k \in \mathbb{Z}_{[N,\infty)}$*. Furthermore, the constraints in* (2) *are satisfied for all time and for all allowable infinite disturbance sequences.*

6.3 Receding Horizon Control Laws

We now consider what happens when the disturbance feedback parameterization (17) is used to design a receding horizon control (RHC) law. In RHC, an admissible feedback policy is computed at each time instant, but only the first component of the policy is applied to the system. An important issue in RHC is whether one can ensure feasibility/constraint satisfaction for all time, despite the fact that a finite horizon is being used and only the first part of the policy is implemented at each sample instant [45].

Consider the *set-valued* map $\kappa_N : X_N^{sf} \to 2^{\mathbb{R}^m}$ ($2^{\mathbb{R}^m}$ is the set of all subsets of $\mathbb{R}^m$), which is defined by considering only the first portion of an admissible state feedback parameter $(\mathbf{L}, \mathbf{g})$, i.e.

$$\kappa_N(x) := \left\{ u \in \mathbb{R}^m \;\middle|\; \exists (\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x) \text{ s.t. } u = L_{0,0}x + g_0 \right\}. \tag{52}$$

An admissible RHC law $\mu_N : X_N^{sf} \to \mathbb{R}^m$ is defined as any selection from the set-valued map $\kappa_N(\cdot)$, i.e. $\mu_N(\cdot)$ has to satisfy

$$\mu_N(x) \in \kappa_N(x), \quad \forall x \in X_N^{sf}. \tag{53}$$

The resulting closed-loop system is then given by

$$x^+ = Ax + B\mu_N(x) + w. \tag{54}$$

Note that the RHC law $\mu_N(\cdot)$ is *time-invariant* and is, in general, a *nonlinear* function of the current state. As discussed in Section 3 it is not always possible to compute an affine or linear RHC law $\mu_N(\cdot)$ off-line such that $\mu_N(x) \in \kappa_N(x)$ for all $x \in X_N^{sf}$. Instead, the selection of an element from $\kappa_N(x)$ has to be done on-line at each sample instant, given a measurement of the current state x. Due to the non-convexity of $\Pi_N^{sf}(x)$, computing an admissible $(\mathbf{L}, \mathbf{g})$ is very difficult. However, by a straightforward application of Theorem 1 it follows that

$$\kappa_N(x) = \left\{ u \in \mathbb{R}^m \;\middle|\; \exists (\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x) \text{ s.t. } u = v_0 \right\}. \tag{55}$$

This fact, combined with the discussion in Section 4 implies that, in order to efficiently compute an admissible RHC input $\mu_N(x)$ at each time instant, one can select an element from $\kappa_N(x)$ by solving a tractable and convex programming problem to obtain an admissible $(\mathbf{M}, \mathbf{v})$ and letting $\mu_N(x) = v_0$.

The following result is easily proven using standard methods in RHC [45], by employing the state feedback parameterization (5):

Proposition 5 (RHC). *If Assumption 1 holds, then the set X_N^{sf} is robust positively invariant for the closed-loop system* (54)*, i.e. if $x \in X_N^{sf}$, then $Ax + B\mu_N(x) + w \in X_N^{sf}$ for all $w \in W$. Furthermore, the constraints* (2) *are satisfied for all time and for all allowable disturbance sequences if and only if the initial state $x(0) \in X_N^{sf}$.*

Obviously, guaranteed invariance of the closed-loop system is not the only desirable result when designing and implementing an RHC law. Section 7 provides some results that allow one to set up a convex and tractable optimization problem, the solution of which can be used to define a *continuous* RHC law $\mu_N(\cdot)$, while also providing a guarantee that the resulting closed-loop system is stable.

6.4 Minimum-time Control Laws

We conclude this section by deriving some results when an admissible affine control policy is used to define a robust minimum-time control law.

Given a maximum horizon length $N_{\max}$ and the set $\mathcal{N} := \{1, \ldots, N_{\max}\}$, let

$$N^*(x) := \min_N \left\{ N \in \mathcal{N} \;\middle|\; \Pi_N^{sf}(x) \neq \emptyset \right\} \tag{56}$$

be the minimum horizon length for which an admissible affine state feedback policy of the form (5) exists. Consider also the set-valued map $\kappa : \mathcal{X} \to 2^{\mathbb{R}^m}$, defined as

$$\kappa(x) := \begin{cases} \kappa_{N^*(x)}(x) & \text{if } x \notin X_f \\ Kx & \text{if } x \in X_f \end{cases} \tag{57}$$

where $\kappa_{N^*(x)}(x)$ is given by (52) with $N = N^*(x)$, $K \in \mathbb{R}^{m \times n}$ and

$$\mathcal{X} := X_f \cup \left(\bigcup_{N \in \mathcal{N}} X_N^{sf} \right). \tag{58}$$

Let the *time-invariant* robust time-optimal control law $\mu : \mathcal{X} \to \mathbb{R}^m$ be any selection from $\kappa(\cdot)$, i.e.

$$\mu(x) \in \kappa(x), \quad \forall x \in \mathcal{X}. \tag{59}$$

Note that $\kappa(\cdot)$ is defined everywhere on $\mathcal{X}$ and that the state of the closed-loop system

$$x^+ = Ax + B\mu(x) + w \tag{60}$$

will enter X_f in less than $N_{\max}$ steps if this is possible, even if Assumption 1 does not hold.

Similar to the discussion in Section 6.3, it is possible to efficiently compute an admissible $\mu(x)$ by recalling (55) and the discussion in Section 4. However, this time it involves solving at most $N_{\max}$ tractable and convex optimization problems at each time step in order to compute $N^*(x)$ and select an element from $\kappa_{N^*(x)}(x)$.

Proposition 6 (Minimum-time control). *If Assumption 1 holds, then $\mathcal{X} = X^{sf}_{N_{\max}}$ and $\mathcal{X}$ is robust positively invariant for the closed-loop system* (60)*, i.e. if $x \in \mathcal{X}$, then $Ax+B\mu(x)+w \in \mathcal{X}$ for all $w \in W$. The state of the closed-loop system enters X_f in $N_{\max}$ steps or less and, once inside, remains inside for all time and for all allowable disturbance sequences. Furthermore, the constraints* (2) *are satisfied for all time and for all allowable disturbance sequences if and only if the initial state $x(0) \in \mathcal{X}$.*

Proof. The proof is straightforward and closely parallels that of Propositions 3 and 5. However, this time one has to show that if $(\mathbf{L}, \mathbf{g})$ is admissible at the current time instant, then a truncated version of $(\mathbf{L}, \mathbf{g})$ is admissible at the next time instant. More precisely, if $(\mathbf{L}, \mathbf{g})$ is admissible at the current time instant and $\underline{\mathbf{L}} := [I_{m(N-1)}\ 0]\mathbf{L}$ and $\underline{\mathbf{g}} := [I_{m(N-1)}\ 0]\mathbf{g}$, then $(\underline{\mathbf{L}}, \underline{\mathbf{g}})$ will be admissible at the next time instant. □

Proposition 6 should be contrasted with Proposition 4. Whereas (51) is a *time-varying* feedback policy that is dependent on current and past values of the state and input, (57) is a *time-invariant* feedback policy that is dependent only on the current state. Note also that (51) does not guarantee that the state of the system will enter X_f in less than $N_{\max}$ steps if this is possible, whereas under the time-optimal control policy $u = \mu(x)$ the state of the system will enter X_f in less than $N_{\max}$ steps if this is possible.

7 Uniqueness, Continuity and Stability of RHC Laws

We next consider the important problem of how to synthesize an RHC law such that the closed-loop system is robustly stable. We choose to minimize the value of a cost function that is quadratic in the disturbance-free states and control inputs and demonstrate that this allows for the synthesis of a continuous control law that guarantees that the closed-loop system is input-to-state stable (ISS). As in Section 6, we rely heavily on Theorem 1 in order to derive these results, choosing to work with whichever of our two parameterizations is most natural in each context.

Before proceeding, we note that alternative cost functions are certainly possible; the reader is referred to [29] where a worst-case quadratic cost function is used and the disturbance is negatively weighted as in $\mathcal{H}_\infty$ control.

7.1 Cost Function

We define an optimal policy pair $(\mathbf{L}^*(x), \mathbf{g}^*(x)) \in \Pi_N^{sf}(x)$ to be one which minimizes the value of a cost function that is quadratic in the disturbance-free state and input sequences. We thus define:

$$V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{w}) := \sum_{i=0}^{N-1} \frac{1}{2}(\|x_i\|_Q^2 + \|u_i\|_R^2) + \frac{1}{2}\|x_N\|_P^2$$

where $\tilde{x}_0 = x$, $\tilde{x}_{i+1} = A\tilde{x}_i + B\tilde{u}_i + w_i$ and $\tilde{u}_i = \sum_{j=0}^{i} L_{i,j}\tilde{x}_j + g_i$ for $i = 0, \ldots, N-1$; the matrices P, Q and R are positive definite, and u_i is given by (5). We define an optimal policy pair as

$$(\mathbf{L}^*(x), \mathbf{g}^*(x)) := \underset{(\mathbf{L},\mathbf{g}) \in \Pi_N^{sf}(x)}{\operatorname{argmin}} \; V_N(x, \mathbf{L}, \mathbf{g}, \mathbf{0}). \tag{61}$$

The receding horizon control policy $\mu_N : X_N^{sf} \to \mathbb{R}^m$ is defined by the first part of the optimal affine state feedback control policy, i.e.

$$\mu_N(x) := L_{0,0}^*(x)x + g_0^*(x) \tag{62}$$

and the closed-loop system then becomes

$$x^+ = Ax + B\mu_N(x) + w. \tag{63}$$

As mentioned in Section 6.3, it is important to note that $\mu_N(\cdot)$ is *nonlinear*, in general, and that it is a *time-invariant* state feedback control law. The difficulty with the control law in (62) is that it is difficult to compute due to the non-convexity of the admissible set $\Pi_N^{sf}(x)$ and the non-convexity of the function $(\mathbf{L}, \mathbf{g}) \mapsto V_N(x, \mathbf{L}, \mathbf{g}, 0)$, hence it is also not obvious whether or not $\mu_N(\cdot)$ is unique or continuous, even if we require $L_{0,0} = 0$[2]. However, by exploiting the equivalent affine disturbance feedback parameterization (17), we will show that the value of the RHC law $\mu_N(x)$, given the current value for x, can be calculated using convex optimization techniques. Furthermore, if $\mathcal{W}$ is a polytope or the affine map of a 1-norm or ∞-norm ball, then the resulting control law $\mu_N(\cdot)$ is unique and continuous.

Remark 9. If the terminal weight P satisfies the discrete algebraic Riccati equation $P = Q + A^TPA - A^TPB(R + B^TPB)^{-1}B^TPA$ and Assumption 1 is satisfied with $K = -(R + B^TPB)^{-1}B^TPA$, then it follows from the principle of optimality that

$$(\mathbf{L}^*(x), \mathbf{g}^*(x)) = \underset{(\mathbf{L},\mathbf{g}) \in \Pi_N^{sf}(x)}{\operatorname{argmin}} \; \frac{1}{2}\sum_{i=0}^{\infty} \hat{x}_i^T Q \hat{x}_i + \hat{u}_i^T R \hat{u}_i, \quad \forall x \in X_N^{sf}, \tag{64}$$

[2]Since the current state x is known, we could have set $L_{0,0} = 0$ without loss of generality. However, the presentation of the results in this paper is simplified by not having this constraint.

where $\hat{x}_0 = x$, $\hat{x}_{i+1} = A\hat{x}_i + B\hat{u}_i$, the control $\hat{u}_i = \sum_{j=0}^{i} L_{i,j}\hat{x}_j + g_i$ for all $i \in \mathbb{Z}_{[0,N-1]}$ and $\hat{u}_i = K\hat{x}_i$ for all $i \in \mathbb{Z}_{[N,\infty)}$. Note also that with this choice of P and K, it follows that

$$\mu_N(x) = Kx, \quad \forall x \in X_f. \tag{65}$$

Before proceeding, we also define the value function $V_N^* : X_N^{sf} \to \mathbb{R}_{\geq 0}$ to be

$$V_N^*(x) := \min_{(\mathbf{L},\mathbf{g})\in\Pi_N^{sf}(x)} V_N(x, \mathbf{L}, \mathbf{g}, 0). \tag{66}$$

We will demonstrate that if $\mathcal{W}$ is a polytope or the affine map of a 1-norm or ∞-norm ball, then the value function $V_N^*(\cdot)$ is Lipschitz continuous, and use it to prove that the resulting closed-loop system is ISS. We will also prove the surprising result that $V_N^*(\cdot)$ is convex, despite the fact that the function $(\mathbf{L}, \mathbf{g}) \mapsto V_N(x, \mathbf{L}, \mathbf{g}, 0)$ is non-convex, in general.

7.2 Exploiting Equivalence to Compute the Value of the RHC Law

For the equivalent affine disturbance feedback parameterization (17), we define a cost function $J_N(\cdot)$ analogous to that defined in (7.1), i.e.

$$J_N(x, \mathbf{M}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}\sum_{i=0}^{N-1}(\bar{x}_i^T Q\bar{x}_i + \bar{u}_i^T R\bar{u}_i) + \frac{1}{2}\bar{x}_N^T P\bar{x}_N \tag{67}$$

where $\bar{x}_0 = x$, $\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i + w_i$ and $\bar{u}_i = \sum_{j=0}^{i-1} M_{i,j}w_j + v_i$ for $i = 0, \ldots, N-1$; the matrices P, Q and R are the same as in (7.1).

If we define

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) := \operatorname*{argmin}_{(\mathbf{M},\mathbf{v})\in\Pi_N^{df}(x)} J_N(x, \mathbf{M}, \mathbf{v}, \mathbf{0}) \tag{68}$$

and

$$\mathbf{v}^*(x) =: \operatorname{vec}(v_0^*(x), \ldots, v_{N-1}^*(x)), \tag{69}$$

then the proof of the following result follows by a straightforward application of Theorem 1.

Proposition 7 (Equivalence for computation of RHC law). *The RHC law $\mu_N(\cdot)$, defined in (62), is given by the first part of the optimal control sequence $\mathbf{v}^*(\cdot)$, i.e.*

$$\mu_N(x) = v_0^*(x) = L_{0,0}^*(x)x + g_0^*(x), \quad \forall x \in X_N^{sf}. \tag{70}$$

The minimum value of $J_N(x, \cdot, \cdot, 0)$ taken over the set of admissible affine disturbance feedback parameters is equal to $V_N^*(x)$, defined in (66), i.e.

$$V_N^*(x) = \min_{(\mathbf{M},\mathbf{v})\in\Pi_N^{df}(x)} J_N(x, \mathbf{M}, \mathbf{v}, 0). \tag{71}$$

Remark 10. Clearly, $(\mathbf{L}^*(x), \mathbf{g}^*(x))$ is found by letting $(\mathbf{M}, \mathbf{v}) = (\mathbf{M}^*(x), \mathbf{v}^*(x))$ in (45).

The above result is important because it allows one to efficiently compute the value of the RHC law given the current state. Together with the discussion in Sections 4.1 and 4.2, the above result implies that for a given $x \in X_N^{sf}$, the value of the RHC law $u = \mu_N(x)$ can be computed via the minimization of a convex function over a convex set. In particular, we remark that if W is the affine map of a 1-norm ball, ∞-norm ball or a polytope, then the optimization problem in (68) can be written as a convex quadratic program (QP) in a tractable number of variables and constraints. If W is an ellipsoid or the affine map of a Euclidean ball, then the optimization problem in (68) becomes a tractable SOCP. In all these cases, the number of decision variables and constraints in the optimization problem is $\mathcal{O}(N^2)$. The reader is referred to [20, 22] for a detailed discussion of an efficient interior point QP implementation that exploits the structure in (68) when the disturbance is ∞-norm bounded.

7.3 Continuity of the RHC Law and Value Function

It is often also a requirement that the control law be unique and continuous in order to avoid undesirable closed-loop behavior, such as chatter. Furthermore, if the value function is continuous, then one can also derive a number of suitable stability results. Hence, we present the following result:

Proposition 8 (Continuity of RHC law and value function). *If W is the affine map of a 1-norm ball, ∞-norm ball or a polytope, then the receding horizon control law $\mu_N(\cdot)$ in (62) is unique and Lipschitz continuous on X_N^{sf}. Furthermore, the value function $V_N^*(\cdot)$ in (66) is strictly convex and Lipschitz continuous on X_N^{sf}.*

Proof. Note that $J_N(x, \mathbf{M}, \mathbf{v}, 0) = J_N(x, 0, \mathbf{v}, 0)$ for all $\mathbf{M}$. Hence, if we define the set

$$\mathcal{V}_N(x) := \left\{ \mathbf{v} \in \mathbb{R}^{Nm} \;\middle|\; \exists \mathbf{M} \text{ such that } (\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x) \right\}, \tag{72}$$

then it follows from (68) that

$$\mathbf{v}^*(x) = \operatorname*{argmin}_{\mathbf{v} \in \mathcal{V}_N(x)} J_N(x, 0, \mathbf{v}, 0) \tag{73}$$

and from Proposition 7, we obtain

$$V_N^*(x) = \min_{\mathbf{v} \in \mathcal{V}_N(x)} J_N(x, 0, \mathbf{v}, 0). \tag{74}$$

Recalling the discussion in Sections 4.1 and 4.2, it is easy to show that if $\mathcal{W}$ is the affine map of a 1-norm ball, ∞-norm ball or a polytope, then $\mathcal{C}_N$, which was defined in (24), is a polyhedron. Since $\mathcal{V}_N(x)$ is the projection of $\mathcal{C}_N$ onto a subspace, $\mathcal{V}_N(x)$ is also a polyhedron and there exist matrices S and T and a vector d such that

$$\mathcal{V}_N(x) = \left\{ \mathbf{v} \in \mathbb{R}^{Nm} \;\mid S\mathbf{v} \leq d + Tx \right\}. \tag{75}$$

It is also easy to verify that

$$J_N(x, 0, \mathbf{v}, 0) = \frac{1}{2}\|\mathbf{B}\mathbf{v} + \mathbf{A}x\|_{\mathcal{Q}}^2 + \frac{1}{2}\|\mathbf{v}\|_{\mathcal{R}}^2, \tag{76}$$

where the block diagonal matrices $\mathcal{Q} \in \mathbb{R}^{nN \times nM}$ and $\mathcal{R} \in \mathbb{R}^{mN \times mN}$ are defined as $\mathcal{Q} := \left[\begin{smallmatrix} I \otimes Q & 0 \\ 0 & P \end{smallmatrix}\right]$ and $\mathcal{R} := I \otimes R$; the block matrices $\mathbf{A}$ and $\mathbf{B}$ are given in the Appendix. Note that $(x, \mathbf{v}) \mapsto J_N(x, 0, \mathbf{v}, 0)$ is a strictly convex quadratic function. Hence, it is straightforward to show that the optimization problems in (73) and (74) are strictly convex QPs of the same structure as in [4]. By applying the results in [4], it follows that $\mathbf{v}^*(\cdot)$ and hence $\mu_N(\cdot)$ are continuous, piecewise affine functions on X_N^{sf} and $V_N^*(\cdot)$ is a strictly convex, piecewise quadratic function on X_N^{sf}. Lipschitz continuity follows from the assumption that $\mathcal{Z}$ is compact, hence X_N^{sf} is also compact. □

Another important property, which will play an important role in the next section in proving stability, is whether the control law vanishes at the origin and whether the minimum of $V^*(\cdot)$ is at the origin. For this purpose, we present the following result:

Lemma 1 (Values at the origin). *If Assumption 1 holds, then* $V_N^*(0) = 0$ *and* $\mu_N(0) = 0$.

Proof. Proposition 3 implies that the origin is in the interior of X_N^{sf}. Note that if $x \in X_f$, then $(\mathbf{L}, \mathbf{g})$ is admissible if $\mathbf{g} = 0$, $L_{i,i} = K$ for $i = 0, \ldots, N-1$ and $L_{i,j} = 0$ for all $i \neq j$. Hence, $V_N^*(0) \leq V_N(0, \mathbf{L}, 0, 0) = 0$. Since $V_N^*(x) \geq 0$ for all $x \in X_N^{sf}$, it follows that $V_N^*(0) = 0$, hence $\mu_N(0) = 0$. □

7.4 Input-to-State Stability (ISS) for RHC

Since the disturbance is non-zero, it is not possible to guarantee that the origin is asymptotically stable, as in conventional RHC without disturbances [45]; without the introduction of further assumptions on the disturbance, the best one can hope for is to guarantee stability and/or attractiveness of a set containing the origin [44]. One way of ensuring robust stability of a set containing the origin is to minimize a suitably-defined worst-case cost [31] or include the current state of the model as an optimization variable [47]. However, since we are minimizing the value of a cost function in the disturbance-free state and control sequences and the current state is fixed, we have to use a different notion of stability. One such alternative is input-to-state stability (ISS), which has already been proven to be effective in the study of RHC laws with input constraints only [25,26,34], RHC laws with discontinuities and output feedback [48] and in the analysis and synthesis of RHC laws with robust constraint satisfaction guarantees [30,39,43].

Consider a nonlinear, time-invariant, discrete-time system of the form

$$x^+ = f(x, w), \tag{77}$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^l$ is a disturbance that takes on values in a compact set $W \subset \mathbb{R}^l$ containing the origin. It is assumed that the state is measured at each time instant, that $f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ is continuous and that $f(0, 0) = 0$. Given a disturbance sequence $w(\cdot)$, where $w(k) \in W$ for all $k \in \mathbb{Z}_{[0,\infty)}$, let the solution to (77) at time k be denoted by $\phi(k, x, w(\cdot))$. For systems of this type, a useful definition of stability is input-to-state stability:

Definition 1 (ISS). For system (77), the origin is *input-to-state stable (ISS)* with region of attraction $\mathcal{X} \subseteq \mathbb{R}^n$, which contains the origin in its interior, if there exist a $\mathcal{KL}$-function $\beta(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that for all initial states $x \in \mathcal{X}$ and disturbance sequences $w(\cdot)$, where $w(k) \in W$ for all $k \in \mathbb{Z}_{[0,\infty)}$, the solution of the system satisfies $\phi(k, x, w(\cdot)) \in \mathcal{X}$ and

$$\|\phi(k, x, w(\cdot))\| \leq \beta(\|x\|, k) + \gamma\left(\sup\left\{\|w(\tau)\| \;\middle|\; \tau \in \mathbb{Z}_{[0,k-1]}\right\}\right) \tag{78}$$

for all $k \in \mathbb{N}$.

Note that input-to-state stability implies that the origin is an asymptotically stable point for the undisturbed system $x^+ = f(x, 0)$ with region of attraction $\mathcal{X}$, and also that all state trajectories are bounded for all bounded disturbance sequences. Furthermore, every trajectory $\phi(x, k, w(\cdot)) \to 0$ if $w(k) \to 0$ as $k \to \infty$. The reader is referred to [28, 33, 54] and the references therein for a thorough treatment of ISS.

In order to be self-contained, we introduce the following useful result, which is easily proven as in [28, Lem 3.5]:

Lemma 2 (ISS-Lyapunov function). *For the system* (77)*, the origin is ISS with region of attraction* $\mathcal{X} \subseteq \mathbb{R}^n$ *if the following conditions are satisfied:*

- $\mathcal{X}$ *contains the origin in its interior and* X_f *is robust positively invariant for* (77)*, i.e.* $f(x, w) \in \mathcal{X}$ *for all* $x \in \mathcal{X}$ *and all* $w \in W$.
- *There exist* $\mathcal{K}_\infty$ *functions* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ *and* $\alpha_3(\cdot)$*, a* $\mathcal{K}$*-function* $\sigma(\cdot)$*, and a continuous function* $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ *such that for all* $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{79a}$$

$$V(f(x, w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) \tag{79b}$$

Remark 11. A function $V(\cdot)$ that satisfies the conditions in Lemma 2 is called an *ISS-Lyapunov function.*

The above result leads immediately to the following, which hints at why Proposition 8 was given:

Lemma 3 (Lipschitz Lyapunov function for undisturbed system). *Let $\mathcal{X} \subseteq \mathbb{R}^n$ contain the origin in its interior and be a robust positively invariant set for* (77)*. Furthermore, let there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ that is Lipschitz continuous on $\mathcal{X}$ such that for all $x \in \mathcal{X}$,*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{80a}$$

$$V(f(x,0)) - V(x) \leq -\alpha_3(\|x\|) \tag{80b}$$

The function $V(\cdot)$ is an ISS-Lyapunov function and the origin is ISS for the system (77) *with region of attraction $\mathcal{X}$ if either of the following conditions are satisfied:*

(i) $f : \mathcal{X} \times W \to \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{X} \times W$.

(ii) $f(x,w) := g(x) + w$, where $g : \mathcal{X} \to \mathbb{R}^n$ is continuous on $\mathcal{X}$.

Proof. Let L_V be the Lipschitz constant of $V(\cdot)$.

(i) Since $\|V(f(x,w)) - V(f(x,0))\| \leq L_V \|f(x,w) - f(x,0)\| \leq L_V L_f \|w\|$, where L_f is the Lipschitz constant of $f(\cdot)$, it follows that $V(f(x,w)) - V(x) = V(f(x,0)) - V(x) + V(f(x,w)) - V(f(x,0)) \leq -\alpha_3(\|x\|) + L_V L_f \|w\|$. The proof is completed by letting $\sigma(s) := L_V L_f s$ in Lemma 2.

(ii) Note that $\|V(f(x,w)) - V(f(x,0))\| \leq L_V \|w\|$. The proof is completed as for (i), but by letting $\sigma(s) := L_V s$ in Lemma 2. □

Remark 12. If $\mathcal{X}$ in Lemmas 2 and 3 is compact, then the condition that $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ be of class $\mathcal{K}_\infty$ can be relaxed to the condition that they only be of class $\mathcal{K}$.

Finally, we add the following assumption, which will allow the value function defined in (66) to be used as an ISS-Lyapunov function:

Assumption 2 (Terminal cost). *The terminal cost $F(x) := x^T P x$ is a Lyapunov function in the terminal set X_f for the undisturbed closed loop system $x^+ = (A + BK)x$ in the sense that*

$$F((A + BK)x) - F(x) \leq -x^T(Q + K^T R K)x, \quad \forall x \in X_f. \tag{81}$$

We can now state our final result:

Theorem 2 (ISS for RHC). *Let W be the affine map of a* 1*-norm ball, ∞-norm ball or a polytope and the RHC law $\mu_N(\cdot)$ be defined as in* (62)*. If Assumptions 1 and 2 hold, then the origin is ISS for the closed-loop system* (63) *with region of attraction X_N^{sf}. Furthermore, the input and state constraints* (2) *are satisfied for all time and for all allowable disturbance sequences if and only if the initial state $x(0) \in X_N^{sf}$.*

Proof. For the system of interest, we of course let $f(x,w) := Ax + B\mu_N(x) + w$. Lemma 1 implies that $f(0,0) = 0$ and Proposition 8 implies that $f(\cdot)$ is continuous on X_N^{sf}.

Combining Proposition 8 with Lemma 1, it follows that $V_N^*(\cdot)$ is a continuous, positive definite function. Hence, there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that (80a) holds with $V(\cdot) := V_N^*(\cdot)$ [33, Lem. 4.3].

Using standard techniques [44, 45], it is easy to show that $V(\cdot) := V_N^*(\cdot)$ is a Lyapunov function for the *undisturbed* system $x^+ = Ax + B\mu_N(x)$. More precisely, the methods in [44,45] can be employed to show that (80b) holds with $\alpha_3(z) := (1/2)\lambda_{\min}(Q)z^2$.

It follows from Proposition 5 that X_N^{sf} is robust positively invariant for system (63). Proposition 3 implies that the origin is in the interior of X_N^{sf}. Finally, recall from Proposition 8 that $\mu_N(\cdot)$ and $V_N^*(\cdot)$ are Lipschitz continuous on X_N^{sf}. By combining all of the above, it follows from Lemma 3 that $V_N^*(\cdot)$ is an ISS-Lyapunov function for system (63). □

Remark 13. Given the same assumptions as in Theorem 2, it can be shown [44, 45] that the origin is an exponentially stable equilibrium (in the classical Lyapunov sense) for the undisturbed system $x^+ = Ax + B\mu_N(x)$ with region of attraction X_N^{sf}.

8 Conclusions

We have shown that the affine state feedback parameterization of Section 3 is equivalent to the affine disturbance feedback parameterization of Section 4. This has the important consequence that, under suitable assumptions on the disturbance and cost function in a given finite horizon optimal control problem, an admissible and optimal state feedback control policy can be found by solving a tractable and convex optimization problem. This is a surprising result, since the set of admissible affine state feedback parameters is non-convex, in general.

In addition, if the optimal control problem involves the minimization of a quadratic cost and the solution is to be implemented in a receding horizon fashion, then one can choose the terminal cost and terminal constraint to guarantee that the closed-loop system is input-to-state stable and that the state and input constraints are satisfied for all time and for all disturbance sequences. Though the conditions to guarantee this are similar to well-known ones in the receding horizon literature, many parts of the proofs of the results are non-standard and rely heavily on the application of the equivalence result of Theorem 1.

A number of open research issues remain to be explored. For example, in this paper it was shown that the proposed disturbance feedback parameterization is equivalent to affine state-feedback with memory. It would be interesting to see if it is possible to derive an equivalent convex re-parameterization in the case where the control at each stage is an affine function of the current state only.

This paper only considered the regulation problem with state feedback. In order to be practically useful, the results in this paper need to be extended to handle the cases of output feedback, setpoint tracking and offset-free control.

It may be possible to extend the continuity and stability results in Proposition 8 and Theorem 2 to cover a broader class of disturbances, such as ellipsoidal or 2-norm bounded disturbances. However, the arguments used for doing so are likely to differ substantially from those given here.

The results in this paper on computational tractability may also be extended to exploit any additional structure inherent in the optimal control problem for specific classes of disturbances and cost functions. Some results along these lines are already available for a class of finite horizon optimal control problems with ∞-norm bounded disturbances [20,22] and the minimization of the finite-horizon ℓ_2 gain of a system [29].

Finally, it is worth mentioning that the results regarding the convexity of the affine disturbance feedback parameterization are easily extended to the case where the disturbance is Gaussian with a known mean and covariance. Since one can no longer guarantee that the set of admissible states is non-empty due to the fact that the disturbance sequence is no longer bounded, the original problem definition has to be changed by requiring that the given state and input constraints only hold with pre-specified probabilities. Methods for converting probabilistic constraints to second-order cone constraints can be found in [12, pp. 157–8] and [55,57]. Once the conversion to second-order cone constraints has been done, an admissible affine disturbance feedback policy can then be found by solving a single, tractable second-order cone program. Though it is possible to set up tractable and convex optimization problems that are equivalent to a class of finite horizon optimal control problems with Gaussian disturbances, a lot of work remains to be done regarding the derivation of meaningful results regarding closed-loop stability and constraint satisfaction.

Acknowledgements

Paul Goulart would like to thank the Gates Cambridge Trust for their support. Eric Kerrigan is a Royal Academy of Engineering Post-doctoral Research Fellow and would like to thank the Royal Academy of Engineering for supporting this research.

References

[1] A. Bemporad. Reducing conservativeness in predictive control of constrained systems with disturbances. In *Proc. 37th IEEE Conf. on Decision and Control*, pages 1384–1391, Tampa, FL, 1998.

[2] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(9):1600–1606, September 2003.

[3] A. Bemporad and M. Morari. *Robust Model Predictive Control: A Survey* in *Robustness in Identification and Control*, volume 245 of *Lecture Notes in Control and Information Sciences*, pages 207–226. Ed. A. Garulli, A. Tesi, and A. Vicino. Springer-Verlag, 1999.

[4] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

[5] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. Technical report, Minerva Optimization Center, Technion, Israeli Institute of Technology, 2002.

[6] D.P. Bertsekas. Infinite-time reachability of state-space regions by using feedback control. *IEEE Trans. Automatic Control*, AC-17(5):604–613, 1972.

[7] D.P. Bertsekas and I.B. Rhodes. On the minimax reachability of target sets and target tubes. *Automatica*, 7(2):233–247, 1971.

[8] D.P. Bertsekas and I.B. Rhodes. Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems. *IEEE Transactions on Automatic Control*, AC-18(2):117–124, April 1973.

[9] F. Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Trans. Automatic Control*, 39(2):428–433, 1994.

[10] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999. Survey paper.

[11] F. Borrelli. *Constrained Optimal Control of Linear and Hybrid Systems*, volume 290 of *Lecture Notes in Control and Information Sciences*. Springer, Germany, 2003.

[12] S. Boyd and L.Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[13] S.P. Boyd and C.H. Barratt. *Linear Controller Design: Limits of Performance*. Prentice Hall, 1991.

[14] E.F. Camacho and C. Bordons. *Model Predictive Control*. Springer, second edition, 2004.

[15] L. Chisci, J.A. Rossiter, and G. Zappa. Systems with persistent state disturbances: predictive control with restricted constraints. *Automatica*, 37:1019–1028, 2001.

[16] M.A. Dahleh and I.J. Diaz-Bobillo. *Control of Uncertain Systems*. Prentice Hall, 1995.

[17] M. Diehl and J. Björnberg. Robust dynamic programming for min-max model predictive control of constrained uncertain systems. *IEEE Transactions on Automatic Control*, 49(12):2253–2257, December 2004.

[18] I.J. Fialho and T.T. Georgiou. ℓ_1 state-feedback control with a prescribed rate of exponential convergence. *IEEE Transactions on Automatic Control*, 42(10):1476–81, October 1997.

[19] S.J. Gartska and R.J.-B Wets. On decision rules in stochastic programming. *Mathematical Programming*, 7:117–143, 1974.

[20] P.J. Goulart and E.C. Kerrigan. An efficient decomposition-based formulation for robust control with constraints. In *16th IFAC World Congress on Automatic Control*, Prague, Czech Repulic, July 2005.

[21] P.J. Goulart and E.C. Kerrigan. Relationships between affine feedback policies for robust control with constraints. In *16th IFAC World Congress on Automatic Control*, Prague, Czech Repulic, July 2005.

[22] P.J. Goulart, E.C. Kerrigan, and D. Ralph. Efficient robust optimization for robust control with constraints. Technical Report CUED/F-INFENG/TR.495, Cambridge University Engineering Department, 2005.

[23] E. Guslitser. Uncertainty-immunized solutions in linear programming. Master's thesis, Technion, Israeli Institute of Technology, June 2002.

[24] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[25] A. Jadbabaie, C. De Persis, and T. Yoon. A globally stabilizing receding horizon controller for neutrally stable linear systems with input constraints. In *Proc. 41st IEEE Conference on Decision and Control*, pages 1096–1100, Las Vegas, Nevada USA, December 2002.

[26] A. Jadbabaie and A. S. Morse. On the ISS property for receding horizon control of constrained linear systems. In *Proc. 15th IFAC World Congress on Automatic Control*, Barcelona, Spain, July 2002.

[27] D. Jia, B.H. Krogh, and O. Stursberg. An LMI approach to robust model predictive control. *Journal of Optimization Theory and Applications*, 2005. In press.

[28] Z. Jiang and Y. Wang. Input-to-state stability for discrete-time non-linear systems. *Automatica*, 37:857–869, 2001.

[29] E.C. Kerrigan and T. Alamo. A convex parameterization for solving constrained min-max problems with a quadratic cost. In *Proc. 2004 American Control Conference*, Boston, MA, USA, June 2004.

[30] E.C. Kerrigan and J.M. Maciejowski. On robust optimization and the optimal control of constrained linear systems with bounded state disturbances. In *Proc. 2003 European Control Conference*, Cambridge, UK, September 2003, 2003.

[31] E.C. Kerrigan and J.M. Maciejowski. Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 14:395–413, 2004.

[32] E.C. Kerrigan and J.M. Maciejowski. Properties of a new parameterization for the control of constrained systems with disturbances. In *Proc. 2004 American Control Conference*, Boston, MA, USA, June 2004.

[33] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, USA, 2002.

[34] J. Kim, T. Yoon, A. Jadbabaie, and C. De Persis. Input-to-state stabilizing MPC for neturally stable linear systems subject to input constraints. In *Proc. 43rd IEEE Conference on Decision and Control*, pages 5041–5046, Paradise Island, Bahamas, December 2004.

[35] I. Kolmanovsky and E.G. Gilbert. Theory and computations of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4:317–363, 1998.

[36] M.V. Kothare, V. Balakrishnan, and M.Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, October 1996.

[37] W. Langson, I. Chryssochoos, S.V. Raković, and D.Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40:125–33, 2004.

[38] Y.I. Lee and B. Kouvaritakis. Constrained receding horizon predictive control for systems with disturbances. *European Journal of Control*, 72(11):1027–1032, August 1999.

[39] D. Limón, T. Álamo, and E.F. Camacho. Robust stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. In *Proc. 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS2004)*, Katholieke Universiteit Leuven, Belgium, July 2004.

[40] J. Löfberg. Approximations of closed-loop MPC. In *Proceedings of the Conference on Decision and Control (CDC)*, pages 1438–1442, Maui, Hawaii, USA, 2003.

[41] J. Löfberg. *Minimax Approaches to Robust Model Predictive Control*. PhD thesis, Linköping University, Apr 2003.

[42] J.M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, UK, 2002.

[43] D. Limón Marruedo, T. Álamo, and E.F. Camacho. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive disturbances. In *Proc. 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada USA, December 2002.

[44] D.Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001. Survey paper.

[45] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[46] D.Q. Mayne and W.R. Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33:2103–2118, 1997.

[47] D.Q. Mayne, M.M. Seron, and S.V. Raković. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41:219–24, 2005.

[48] M.J. Messina, S.E. Tuna, and A.R. Teel. Discrete-time certainty equivalence output feedback: allowing discontinuous control laws including those from model predictive control. *Automatica*, 41(4):617–628, 2005.

[49] S.V. Rakovic, E.C. Kerrigan, K.I. Kouramas, and D.Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Trans. Automatic Control*, 50(3), March 2005.

[50] D. Ramirez and E.F. Camacho. On the piecewise linear nature of constrained min-max model predictive control with bounded uncertainties. In *Proc. 2003 American Control Conference*, volume 3, pages 2453–8, Denver CO, USA, June 2003.

[51] P.O.M. Scokaert and D.Q. Mayne. Min-max feedback model predictive control for constrained linear systems. *IEEE Trans. Automatic Control*, 43:1136–1142, 1998.

[52] J.S. Shamma. Optimization of the ℓ_∞-induced norm under full state feedback. *IEEE Transactions on Automatic Control*, 41(4):533–44, April 1996.

[53] R.S. Smith. Robust model predictive control of constrained linear systems. In *Proc. 2004 American Control Conference*, pages 245–250, Boston MA, USA, June 2004.

[54] E.D. Sontag. The ISS philosophy as a unifying framework for stability-like behavior. In A. Isidori *et al*, editor, *Nonlinear Control in the Year 2000 (Volume 2)*, Lecture Notes in Control and Information Sciences, pages 443–468. Springer, Germany, 2000.

[55] D.H. van Hessem. *Stochastic inequality constrained closed-loop model predictive control with application to chemical process operation.* PhD thesis, Technical University of Delft, June 2004.

[56] D.H. van Hessem and O.H. Bosgra. A conic reformulation of model predictive control including bounded and stochastic disturbances under state and input constraints. In *41st IEEE Conference on Decision and Control*, pages 4643–4648, December 2002.

[57] D.H. van Hessem, C.W. Scherer, and O.H. Bosgra. LMI-based closed-loop economic optimization of stochastic process operation under state and input constraints. In *Proc. 40th IEEE Conference on Decision and Control*, Orlando FL, USA, December 2001.

[58] H.S. Witsenhausen. A minimax control linear problem for sampled systems. *IEEE Transactions on Automatic Control*, AC-13(1):5–21, 1968.

Appendix

Let the matrices $\mathbf{A} \in \mathbb{R}^{n(N+1)\times n}$ and $\mathbf{E} \in \mathbb{R}^{n(N+1)\times nN}$ be defined as

$$\mathbf{A} := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{E} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ A & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I_n \end{bmatrix}. \tag{82}$$

We also define the matrices $\mathbf{B} \in \mathbb{R}^{n(N+1)\times mN}$, $\mathbf{C} \in \mathbb{R}^{t\times n(N+1)}$ and $\mathbf{D} \in \mathbb{R}^{t\times mN}$ as

$$\mathbf{B} := \mathbf{E}(I_N \otimes B),\ \mathbf{C} := \begin{bmatrix} I_N \otimes C & 0 \\ 0 & Y \end{bmatrix},\ \mathbf{D} := \begin{bmatrix} I_N \otimes D \\ 0 \end{bmatrix}. \tag{83}$$

It is easy to check that (21) is equivalent to (23) with

$$F := \mathbf{CB} + \mathbf{D},\ G := \mathbf{CE},\ H := -\mathbf{CA},\ c := \begin{bmatrix} \mathbf{1}_N \otimes b \\ z \end{bmatrix}. \tag{84}$$