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Abstract

This paper addresses the design of a dynamic, nonlinear, time-invariant, state feedback controller that guarantees constraint satisfaction and offset-free control in the presence of unmeasured, persistent, non-stationary, additive disturbances. First, this objective is obtained by designing a dynamic, linear, time-invariant, offset-free controller, and an appropriate domain of attraction for this linear controller is defined. Following this, the linear (unconstrained) control input is modified by adding a perturbation term that is computed by a robust receding horizon controller. It is shown that the domain of attraction of the receding horizon controller contains that of the linear controller, and an efficient implementation of the receding horizon controller is proposed. Proofs of robust constraint satisfaction and offset-free control are given, and the effectiveness of the proposed controller is illustrated through an example of a continuous stirred tank reactor.

Key words: Integral control, receding horizon control, set invariance, dynamic state feedback control, nonlinear control, constrained systems.

1 Introduction

The control of systems in the presence of constraints is an important task in many application fields because constraints "always" arise from physical limitations and quality or safety reasons. Moreover, in practical applications disturbances are usually present, and often they are not measurable and predictable. For example, in the chemical industries disturbances arise from interactions between different plant units, from changes in the raw materials and in the operating conditions (such as ambient temperature, humidity, etc.).

The design of control algorithms able to stabilize plants subject to unknown bounded disturbances in the presence of input and state constraints has been the subject of several works [1, 2, 3, 4]. A number of surveys are available [5, 6, 7], which discuss how the important goal of guaranteeing closed-loop stability and constraint satisfaction can be obtained.

In many practical applications, especially in the process industries, disturbances are often non-stationary. It is clear that if an unmeasured disturbance keeps changing with time, offset-free control is not possible,

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whereas if the disturbance is non-stationary (i.e. integrating), offset-free control is an achievable goal. One basic objective of an effective control algorithm is that it guarantees offset-free control whenever this is possible.

However, none of the existing algorithms with stability guarantees can also guarantee offset-free control in the case of non-stationary disturbances. In this paper, a novel control design method for constrained systems subject to unmeasured bounded disturbance is presented. The proposed controller is guaranteed to remove steady-state offset in the controlled variables whenever the disturbance reaches an (unknown) constant value. The controller is also guaranteed to satisfy input and state constraints.

This paper is organized as follows. In Section 2 the problem definition is given and in Section 3 the design of a linear offset-free controller is presented along with detailed discussions about its closed-loop properties and its domain of attraction. In Section 4 a nonlinear controller is designed, using ideas from model predictive control, in order to enlarge the domain of attraction. The main characteristics are illustrated in Section 5 through an example of a continuous stirred tank reactor. Finally, the main achievements of this work are summarized in Section 6. Proofs and additional definitions are given in Appendix A and B, respectively.

Notation: If a and b are column vectors, then (a,b) will be used to denote the column vector $[a^T \ b^T]^T$. Given two matrices A and B, the Kronecker product is denoted by $A \otimes B$. The set of non-negative integers is $\mathbb{N} := \{0,1,2,\dots\}$. Where it will not lead to confusion, $\omega(k)$ will denote the *actual* value of the infinite sequence $\omega(\cdot)$ at time k, while ω_k will be used to denote the *prediction* of $\omega(\tau+k)$ at a time instant k steps into the future if $\omega = \omega_0 = \omega(\tau)$ is the value of the variable at current time τ. Given a set Ω, $\mathscr{M}_\Omega$ is the set of infinite sequences $\omega(\cdot) := \{\omega(0), \omega(1), \dots\}$ that take on values in Ω, i.e. $\mathscr{M}_\Omega := \{\omega(\cdot) \mid \omega(k) \in \Omega, \forall k \in \mathbb{N}\}$. If the set $\Omega \subset X \times Y$, then $\text{Proj}_X(\Omega) := \{x \in X \mid \exists y \in Y \text{ such that } (x,y) \in \Omega\}$ is the projection of Ω onto X. Given a positive integer N, I_N is the identity matrix with N rows and N columns, $\mathbf{1}_N$ is a column vector of ones of length N and the Cartesian product $\Omega^N := \underbrace{\Omega \times \cdots \times \Omega}_{N \text{ times}}$.

2 Problem Description

In this paper we consider a discrete-time, linear, time-invariant plant:

$$x^+ = Ax + Bu + Ed\,, \tag{1a}$$

$$z = C_z x\,, \tag{1b}$$

in which $x \in \mathbb{R}^n$ is the plant state, x^+ is the plant successor state, $u \in \mathbb{R}^m$ is the control input (manipulated variable), $d \in \mathbb{R}^r$ is a persistent, unmeasured disturbance and $z \in \mathbb{R}^p$ is the controlled variable, i.e. the variable to be controlled to the origin. Affine inequality constraints are given on the state and input, i.e.

$$x \in \mathscr{X} \subset X\,, \quad u \in \mathscr{U} \subset U\,, \tag{2}$$

where $X := \mathbb{R}^n$ is the state space, $U := \mathbb{R}^m$ is the input space, $\mathscr{X}$ is a polyhedron (a closed and convex set that can be described by a finite number of affine inequality constraints) and $\mathscr{U}$ is a polytope (a bounded polyhedron); the origin is contained in the interior of $\mathscr{X} \times \mathscr{U}$.

Assumption 1 (General). A measurement of the plant state is available at each sample instant, (A,B) is stabilizable, (A,C_z) is detectable and

$$\text{rank} \begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix} = n + p\,. \tag{3}$$

Notice that the last condition implies that the dimension of the controlled variable cannot exceed the dimension of either the state or the input, i.e. $p \leq \min\{n,m\}$.

A *dynamic*, nonlinear, time-invariant state feedback controller is to be designed and is to assume the following structure:

$$\sigma^+ = \alpha(x,\sigma)\,, \tag{4a}$$

$$u = \gamma(x,\sigma)\,, \tag{4b}$$

where $\sigma \in \mathbb{R}^l$ is the controller state, σ^+ is the controller successor state, $\alpha : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l$ is the controller state dynamics map and $\gamma : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m$ is the controller output map.

Remark 1. In this paper both $\alpha(\cdot)$ and $\gamma(\cdot)$ will be nonlinear.

The plant dynamics (1a), together with the controller (4), forms a closed-loop system

$$\xi^+ = f(\xi,d)\,, \tag{5}$$

where

$$\xi := \begin{bmatrix} x \\ \sigma \end{bmatrix} \tag{6}$$

is the closed-loop system state and the closed-loop dynamics are given by

$$f(\xi,d) := \begin{bmatrix} Ax + B\gamma(x,\sigma) \\ \alpha(x,\sigma) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d\,. \tag{7}$$

Let $\phi(k,\xi,d(\cdot))$ be the solution to (5) at time k when the augmented state is ξ at time 0 (note that since the system is time-invariant, the current time can always be regarded as zero) and the disturbance sequence is $d(\cdot) := \{d(k)\}_{k=0}^{\infty}$, i.e.

$$\phi(k,\xi,d(\cdot)) := \underbrace{f(f(\ldots(f}_{k \text{ times}}(\xi,d(0)),d(1))\ldots),d(k-1))\,.$$

By definition, $\phi(0,\xi,d(\cdot)) := \xi$. With a slight abuse of notation, we also define the following:

$$\xi(k) := \phi(k,\xi,d(\cdot))\,, \tag{8a}$$

$$x(k) := \begin{bmatrix} I_n & 0 \end{bmatrix} \phi(k,\xi,d(\cdot))\,, \tag{8b}$$

$$\sigma(k) := \begin{bmatrix} 0 & I_l \end{bmatrix} \phi(k,\xi,d(\cdot))\,, \tag{8c}$$

$$u(k) := \gamma(\phi(k,\xi,d(\cdot)))\,, \tag{8d}$$

$$z(k) := \begin{bmatrix} C_z & 0 \end{bmatrix} \phi(k,\xi,d(\cdot))\,. \tag{8e}$$

Given a controller defined in (4) and an infinite disturbance sequence $d(\cdot)$, the resulting closed-loop trajectories of the individual variables are then denoted by $\{\xi(k)\}_{k=0}^{\infty}$, $\{x(k)\}_{k=0}^{\infty}$, $\{\sigma(k)\}_{k=0}^{\infty}$, $\{u(k)\}_{k=0}^{\infty}$ and $\{z(k)\}_{k=0}^{\infty}$.

In general, since the disturbance is persistent and unknown it is impossible to drive the controlled variable to the origin. However, we consider the following restriction on the disturbance:

Assumption 2 (Disturbance). At each time instant, the current and future disturbances are unknown. The disturbance sequence $d(\cdot)$ takes on values in a polytope $\mathscr{D} \subset \mathbb{R}^r$ containing the origin and asymptotically reaches an unknown steady-state value, i.e. $d(k) \in \mathscr{D}$ for all $k \in \mathbb{N}$ and there exists a $\bar{d} \in \mathscr{D}$ such that $\lim_{k\to\infty} d(k) = \bar{d}$.

Under the above assumptions we present a novel method for designing a dynamic, nonlinear, time-invariant state feedback controller (4) that, for any allowable disturbance sequence (any infinite disturbance sequence that satisfies Assumption 2), accomplishes the goal of driving the controlled variable to the origin, while respecting the state and input constraints, i.e.

$$\lim_{k\to\infty} z(k) = 0 \tag{9a}$$

and

$$x(k) \in \mathscr{X}, \quad u(k) \in \mathscr{U} \tag{9b}$$

for all $d(\cdot) \in \mathscr{M}_{\mathscr{D}}$ and all $k \in \mathbb{N}$.

3 Linear Controller Design

3.1 The Augmented System

In order to address the problem we make use of the following auxiliary system to define the controller state dynamics:

$$\hat{x}^+ = Ax + Bu + (\hat{d} + x - \hat{x}), \tag{10a}$$

$$\hat{d}^+ = \hat{d} + x - \hat{x}. \tag{10b}$$

Remark 2. The system (10) corresponds to using a dead-beat observer for the following system:

$$\begin{bmatrix}\hat{x}\\ \hat{d}\end{bmatrix}^+ = \begin{bmatrix}A & I\\ 0 & I\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{d}\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix} u,$$

$$x = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{d}\end{bmatrix},$$

in which it is clear that $\hat{d} \in \mathbb{R}^n$ is an integrating (step) disturbance acting on the state $\hat{x} \in \mathbb{R}^n$. The role of $\hat{d}$ is essential in removing steady-state offset in the presence of an unknown persistent disturbance [8, 9] and will be clarified later. As will be seen later, the dimensions of $\hat{d}$ and d need not be the same in order to guarantee offset-free control.

By combining the plant dynamics (1a) and the auxiliary system (10), we obtain the following augmented system:

$$\xi^+ = \mathscr{A}\xi + \mathscr{B}u + \mathscr{E}d, \tag{11}$$

in which

$$\xi := \begin{bmatrix}x\\ \hat{x}\\ \hat{d}\end{bmatrix}, \; \mathscr{A} := \begin{bmatrix}A & 0 & 0\\ I+A & -I & I\\ I & -I & I\end{bmatrix}, \; \mathscr{B} := \begin{bmatrix}B\\ B\\ 0\end{bmatrix}, \; \mathscr{E} := \begin{bmatrix}E\\ 0\\ 0\end{bmatrix}. \tag{12}$$

We also define the controller state $\sigma \in \mathbb{R}^l$, with $l := 2n$, to be the states of the auxiliary system (10), i.e.

$$\sigma := \begin{bmatrix}\hat{x}\\ \hat{d}\end{bmatrix}. \tag{13}$$

3.2 Unconstrained Offset-free Controller Design

When a non-zero persistent disturbance affects a system, the origin of the state and input needs to be shifted in order to cancel the effect of such a disturbance on the controlled variable [10, 11]. To this aim, at each sample instant we use the estimate of the future disturbance and compute the steady-state target $(\bar{x},\bar{u})$ such that one can drive the controlled variable to the origin. When the dimension of the input is equal to the dimension of the controlled variable ($m = p$) these targets are uniquely defined by:

$$\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} \hat{d}^+ \\ 0 \end{bmatrix} = \begin{bmatrix} I & -I & I \\ 0 & 0 & 0 \end{bmatrix} \xi\,. \tag{14}$$

Notice that this corresponds to finding the pair $(\bar{x},\bar{u})$ such that $C_z\bar{x} = 0$ and $\bar{x} = A\bar{x} + B\bar{u} + \hat{d}^+$, i.e. the state and input that cancel the effect of the disturbance. If, instead, there are extra degrees of freedom ($m > p$) these targets are non-unique. However, one can address both cases [11] by solving the following equality-constrained quadratic program, in which $\bar{R} \in \mathbb{R}^{m\times m}$ is a positive definite matrix:

$$(\bar{x}^*(\xi), \bar{u}^*(\xi)) := \arg\min_{(\bar{x},\bar{u})} \frac{1}{2}\bar{u}^T \bar{R} \bar{u}, \tag{15a}$$

subject to

$$\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} I & -I & I \\ 0 & 0 & 0 \end{bmatrix} \xi\,. \tag{15b}$$

For a given augmented state ξ, one can think of $(\bar{x}^*(\xi), \bar{u}^*(\xi))$ as the new 'origin' around which the system should be regulated. Solving for $(\bar{x}^*(\xi), \bar{u}^*(\xi))$ is trivial:

Lemma 1 (Target calculation). *The minimizer of the equality-constrained quadratic program* (15) *is linear with respect to the augmented state ξ and is given by*

$$\begin{bmatrix} \bar{x}^*(\xi) \\ \bar{u}^*(\xi) \end{bmatrix} = \begin{bmatrix} \Pi_{13} & -\Pi_{13} & \Pi_{13} \\ \Pi_{23} & -\Pi_{23} & \Pi_{23} \end{bmatrix} \xi, \tag{16}$$

where $\Pi_{13} \in \mathbb{R}^{n\times n}$ and $\Pi_{23} \in \mathbb{R}^{m\times n}$ are the relevant block matrix components of

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{34} \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} \end{bmatrix} := \begin{bmatrix} 0 & 0 & -I+A^T & -C_z^T \\ 0 & \bar{R} & B^T & 0 \\ I-A & -B & 0 & 0 \\ C_z & 0 & 0 & 0 \end{bmatrix}^{-1} \tag{17}$$

and $\begin{bmatrix} \Pi_{11} & \Pi_{12} \end{bmatrix}$ has $m+n$ columns.

Proof. See Appendix A.1. □

We now consider what would happen if one were to choose a gain matrix K such that $A+BK$ is strictly stable and let the control input in the augmented system (11) be given by

$$u = \bar{u}^*(\xi) + K(x - \bar{x}^*(\xi))\,. \tag{18}$$

Remark 3. It is straightforward [10] to show that if K is computed through an appropriate Riccati equation, then the control law defined by (18) corresponds to the solution of the following LQR problem, in which Q and R are positive definite matrices of appropriate dimension:

$$\min_{\{u_k\}_{k=0}^{\infty}} \frac{1}{2} \sum_{k=0}^{\infty} (x_k - \bar{x}^*(\xi))^T Q (x_k - \bar{x}^*(\xi)) + (u_k - \bar{u}^*(\xi))^T R (u_k - \bar{u}^*(\xi))\,,$$

subject to $x_0 = x$ and

$$x_{k+1} - \overline{x}^*(\xi) = A(x_k - \overline{x}^*(\xi)) + B(u_k - \overline{u}^*(\xi)), \quad \forall k \in \mathbb{N}.$$

Before proceeding, we need the following result:

Lemma 2 (Stability). *Suppose that Assumption 1 holds and $K \in \mathbb{R}^{m\times n}$ is such that $A+BK$ is strictly stable. If $\mathscr{A}$ and $\mathscr{B}$ are given by* (12), $\Gamma \in \mathbb{R}^{m\times n}$ *is any constant matrix and*

$$\mathscr{K} := \begin{bmatrix} K+\Gamma & -\Gamma & \Gamma \end{bmatrix}, \tag{19}$$

then

$$\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K} \tag{20}$$

is strictly stable.

Proof. See Appendix A.2. □

By defining

$$\Gamma := \Pi_{23} - K\Pi_{13}, \tag{21}$$

and substituting (16) into (18) it follows that

$$u = \Pi_{23}(x - \hat{x} + \hat{d}) + K(x - \Pi_{13}(x - \hat{x} + \hat{d})) \tag{22a}$$
$$= (K+\Gamma)x - \Gamma\hat{x} + \Gamma\hat{d} \tag{22b}$$
$$= \mathscr{K}\xi. \tag{22c}$$

After substituting (22) into (11), one can write an expression for the augmented system (11) under the linear control $u = \mathscr{K}\xi$ as

$$\xi^+ = \mathscr{A}_{\mathscr{K}}\xi + \mathscr{E}d. \tag{23}$$

Let $\psi(k, \xi, d(\cdot))$ be the solution of the closed-loop system (23) at time k, given the state ξ at time 0 and the disturbance sequence $d(\cdot)$.

As a consequence of the above, we introduce the following standing assumption:

Assumption 3 (Stabilizing gain). The matrix $K \in \mathbb{R}^{m\times n}$ is chosen such that $A+BK$ is strictly stable, $\mathscr{K}$ is given by (19) with Γ given by (21) and $\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K}$.

The following result states that if the control is given by $u = \mathscr{K}\xi$, then the value of the controlled variable for (23) is guaranteed to converge to the origin, given any allowable infinite disturbance sequence:

Lemma 3 (Offset-free control). *If Assumptions 1–3 hold, then the closed-loop system* (23) *satisfies*

$$\lim_{k\to\infty} \begin{bmatrix} C_z & 0 \end{bmatrix} \psi(k, \xi, d(\cdot)) = 0. \tag{24}$$

for all $\xi \in \mathbb{R}^{3n}$ and all $d(\cdot) \in \mathscr{M}_{\mathscr{D}}$.

Proof. See Appendix A.3. □

3.3 The Maximal Constraint-Admissible Robustly Positively Invariant Set

We now consider the problem of computing the maximal constraint-admissible robustly positively invariant set in the space of the augmented state $\xi := [x^T \ \hat{x}^T \ \hat{d}^T]^T$.

Let the *constraint-admissible set* Ξ be defined as

$$\Xi := \left\{ \xi \in \mathbb{R}^{3n} \, | x \in \mathscr{X} \text{ and } \mathscr{K}\xi \in \mathscr{U} \right\} . \tag{25}$$

The *maximal constraint-admissible robustly positively invariant set* $\mathscr{O}_\infty$ for the closed-loop system (23) is defined as all initial states in Ξ for which the evolution of the system remains in Ξ for all allowable infinite disturbance sequences:

$$\mathscr{O}_\infty := \{ \xi \in \Xi \mid \psi(k,\xi,d(\cdot)) \in \Xi, \, \forall d(\cdot) \in \mathscr{M}_\mathscr{D}, \forall k \in \mathbb{N} \} . \tag{26}$$

Assumption 4 (Invariant set). The set $\mathscr{O}_\infty$ as defined in (26) is non-empty, contains the origin in its interior and is finitely determined (described by a finite number of affine inequality constraints).

Since (23) is linear and time-invariant and Ξ is given by a finite number of affine inequality constraints, $\mathscr{O}_\infty$ is easily computed by solving a finite number of LPs [12].

Remark 4. Except for a few pathological cases, Assumption 4 is met if $\mathscr{A}_\mathscr{K}$ is strictly stable, $\mathscr{X}$ is bounded, $([I_n \ 0], \mathscr{A}_\mathscr{K})$ is observable and $\mathscr{D}$ is sufficiently small [12]; however, observability of $([I_n \ 0], \mathscr{A}_\mathscr{K})$ and boundedness of $\mathscr{X}$ are not guaranteed under the assumptions in this paper. Despite this, in all test cases we have found that Assumption 4 holds. If Assumption 4 is violated, it is easy to modify the problem such that it is satisfied, e.g. by intersecting Ξ or $\mathscr{X}$ with a sufficiently large bounded polyhedron. The reader is referred to [12] for alternative modifications that guarantee that Assumption 4 holds.

The following result states that, provided the augmented state is in $\mathscr{O}_\infty$ at time 0, then the evolution of the augmented system under the linear control $u = \mathscr{K}\xi$ is such that offset-free control is guaranteed and the state and input constraints are satisfied for all allowable disturbance sequences:

Proposition 1 (Linear controller). *Suppose that Assumptions 1–4 hold. The solution of the closed-loop system* (23) *satisfies* (24) *and*

$$\begin{bmatrix} I_n & 0 \end{bmatrix} \psi(k,\xi,d(\cdot)) \in \mathscr{X} \text{ and } \mathscr{K}\psi(k,\xi,d(\cdot)) \in \mathscr{U} , \tag{27}$$

for all $\xi \in \mathscr{O}_\infty$, all $d(\cdot) \in \mathscr{M}_\mathscr{D}$ and all $k \in \mathbb{N}$.

Proof. The result follows immediately from the discussion above and the proof is based on the invariance of $\mathscr{O}_\infty$ for the closed-loop system (23) and the fact that $\mathscr{O}_\infty$ is constraint-admissible. □

Because of the assumptions in Proposition 1, it is important to initialize the controller state $\sigma := [\hat{x}^T \ \hat{d}^T]^T$ correctly such that $\xi := [x^T \ \sigma^T]^T \in \mathscr{O}_\infty$ at time 0. A sensible way to initialize the controller state is to compute the minimizer of the following quadratic program, given the initial plant state $x(0)$:

$$\left(\hat{x}(0), \hat{d}(0)\right) := \arg\min_{(\hat{x},\hat{d})} \left\{ (x(0)-\hat{x})^T (x(0)-\hat{x}) + \hat{d}^T \hat{d} \;\middle|\; \xi \in \mathscr{O}_\infty \right\} . \tag{28}$$

We can now also define X_0 to be the set of plant states for which there exists a controller state such that the augmented state is in $\mathscr{O}_\infty$:

$$X_0 := \left\{ x \in \mathbb{R}^n \;\middle|\; \exists \sigma \in \mathbb{R}^{2n} \text{ such that } \xi \in \mathscr{O}_\infty \right\} . \tag{29}$$

Clearly, (28) is feasible if and only if $x(0) \in X_0$. Note that since $\mathscr{O}_\infty$ is a polyhedron, the set X_0 can be computed as the projection [13, 14] of $\mathscr{O}_\infty$ onto the plant state space X:

$$X_0 = \mathrm{Proj}_X \left(\mathscr{O}_\infty\right) . \tag{30}$$

4 Receding Horizon Controller Design

The set X_0 is the set of initial plant states for which the controlled variable will be driven to the origin by the linear control $u = \mathscr{K}\xi$. This section presents an efficient approach for computing a nonlinear controller, which enlarges the set of initial plant states for which the controlled variable can ultimately be driven to the origin. This will be achieved by using ideas from model predictive control for constrained systems [6, 7, 15].

4.1 Definition and Properties of the Receding Horizon Controller

Similar to the idea proposed in [3, 16] of 'pre-stabilizing' the plant, let the linear control in (22) be modified with a perturbation term as follows:

$$u = \mathscr{K}\xi + v, \tag{31}$$

where $v \in \mathbb{R}^m$ is the input perturbation. The solution to the finite horizon optimal control problem (FHOCP), defined below, is a finite sequence of input perturbations that guarantees robust constraint satisfaction over the horizon and optimizes some cost function. Under the control (31) the augmented state dynamics (11) become

$$\xi^+ = \mathscr{A}_{\mathscr{K}}\xi + \mathscr{B}v + \mathscr{E}d. \tag{32}$$

Before proceeding, let the horizon length N be a positive integer and the block vectors $\mathbf{v} \in \mathbb{R}^{mN}$ and $\mathbf{d} \in \mathbb{R}^{rN}$ be defined as

$$\mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad \mathbf{d} := \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{N-1} \end{bmatrix}, \tag{33}$$

where $v_k \in \mathbb{R}^m$ and $d_k \in \mathbb{R}^r$ for all $k \in \{0, \dots, N-1\}$.

With a slight abuse of notation, let

$$\xi_k := \chi(k, \xi, \mathbf{v}, \mathbf{d}) := \begin{cases} \mathscr{A}_{\mathscr{K}}^k \xi + \sum_{i=0}^{k-1} \mathscr{A}_{\mathscr{K}}^i (\mathscr{B}v_{k-1-i} + \mathscr{E}d_{k-1-i}) & \text{if } k > 0 \\ \xi & \text{if } k = 0 \end{cases} \tag{34}$$

denote the solution to (32) for all $k \in \{0, \dots, N\}$, given the augmented state ξ, a sequence of control perturbations $\mathbf{v}$ and a sequence of disturbances $\mathbf{d}$. The corresponding predicted plant state and input are similarly defined as

$$x_k := \begin{bmatrix} I_n & 0 \end{bmatrix} \chi(k, \xi, \mathbf{v}, \mathbf{d}), \qquad \forall k \in \{0, \dots, N\}, \tag{35a}$$

$$u_k := \mathscr{K}\chi(k, \xi, \mathbf{v}, \mathbf{d}) + v_k, \qquad \forall k \in \{0, \dots, N-1\}. \tag{35b}$$

The set of admissible input perturbations $\mathscr{V}_N(\xi)$ is the set of input perturbations of length N such that for all allowable disturbances of length N, the input constraints $\mathscr{U}$ are satisfied over the horizon $k = 0, \dots, N-1$, the state constraints $\mathscr{X}$ are satisfied over the horizon $k = 1, \dots, N-1$ and the augmented state at the end of the horizon is in $\mathscr{O}_\infty$ (hence the predicted plant state at the end of the horizon is also in $\mathscr{X}$):

$$\mathscr{V}_N(\xi) := \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; \begin{array}{r} \xi_0 = \xi,\ x_k \in \mathscr{X},\ k = 1, \dots, N-1,\ \xi_N \in \mathscr{O}_\infty \text{ and} \\ u_k \in \mathscr{U},\ k = 0, \dots, N-1 \text{ for all } \mathbf{d} \in \mathscr{D}^N \end{array} \right\}. \tag{36}$$

Remark 5. Note that $\mathscr{V}_N(\xi)$ is defined by an *infinite* number of constraints. Obtaining an equivalent expression for $\mathscr{V}_N(\xi)$ in terms of a *finite* number of affine inequality constraints is straightforward and a result that allows one to do this efficiently is given in Section 4.2.

In order to define the receding horizon controller, we need to define an associated FHOCP. Similar to [3, 16], we choose to define $\mathbb{P}_N(\xi)$, the FHOCP to be solved for a given ξ, as

$$\mathbb{P}_N(\xi): \quad J_N^*(\xi) := \min_{\mathbf{v}} \{ J_N(\mathbf{v}) \mid \mathbf{v} \in \mathscr{V}_N(\xi) \} , \tag{37a}$$

where the cost function to be minimized is defined as

$$J_N(\mathbf{v}) := \sum_{k=0}^{N-1} v_k^T W v_k , \tag{37b}$$

in which W is a positive definite matrix. The minimizer of $\mathbb{P}_N(\xi)$ is similarly defined:

$$\mathbf{v}^*(\xi) := \left(v_0^*(\xi), \ldots, v_{N-1}^*(\xi) \right) := \arg\min_{\mathbf{v}} \{ J_N(\mathbf{v}) \mid \mathbf{v} \in \mathscr{V}_N(\xi) \} . \tag{37c}$$

We assume here that the minimizer of $\mathbb{P}_N(\xi)$ exists; this assumption is justified in Section 4.2.

As is standard in receding horizon control [6, 7, 15], for a given state ξ, we only keep the first element $v_0^*(\xi)$ of the solution to the FHOCP. Using this receding horizon principle, we define our controller in (4) by substituting

$$u = \mathscr{K}\xi + v_0^*(\xi) \tag{38}$$

into the equation for the augmented system (11) and comparing it with the expression for the closed-loop dynamics (7). In other words, the controller state dynamics map in (4a) is given by

$$\alpha(x,\sigma) := \begin{bmatrix} I+A & -I & I \\ I & -I & I \end{bmatrix} \xi + \begin{bmatrix} B\mathscr{K} \\ 0 \end{bmatrix} \xi + \begin{bmatrix} B \\ 0 \end{bmatrix} v_0^*(\xi) \tag{39a}$$

and the controller output map in (4b) is

$$\gamma(x,\sigma) := \mathscr{K}\xi + v_0^*(\xi) . \tag{39b}$$

It is important to be able to determine all the plant states for which one can guarantee that problem $\mathbb{P}_N(\xi)$ has a solution. The set of plant states $X_N^{\mathbf{v}}$ for which one can initialize the controller state such that the set of admissible input perturbations $\mathscr{V}_N(\xi)$ is non-empty (and $\mathbb{P}_N(\xi)$ has a solution) is given by

$$X_N^{\mathbf{v}} := \left\{ x \in \mathscr{X} \;\middle|\; \exists \sigma \in \mathbb{R}^{2n} \text{ such that } \mathscr{V}_N(\xi) \neq \emptyset \right\} . \tag{40}$$

As will be shown below, $X_N^{\mathbf{v}}$ is the set of plant states in $\mathscr{X}$ for which the controlled variable will be driven to the origin by the controller (4), if α and γ are given by (39).

We can now give our first main result:

Theorem 1 (Domain of RHC). *Suppose that Assumptions 1–4 hold. The sequence of sets* $\{X_0, X_1^{\mathbf{v}}, \ldots, X_N^{\mathbf{v}}\}$, *where* X_0 *is defined in* (29) *and each* $X_i^{\mathbf{v}}$, $i \in \{1,\ldots,N\}$, *is defined as in* (40) *with* $N = i$, *contains the origin in their interiors and satisfies the set inclusion*

$$X_0 \subseteq X_1^{\mathbf{v}} \subseteq \cdots \subseteq X_{N-1}^{\mathbf{v}} \subseteq X_N^{\mathbf{v}} . \tag{41}$$

Proof. See Appendix A.4. □

Theorem 1 is very important because it shows that, under the above assumptions, an increase in the horizon length does not decrease the size of the set of plant states for which the controlled variable can be driven to the origin.

Before giving our second main result, we need the following:

Lemma 4 (Perturbation sequence). *Suppose that Assumptions 1–4 hold. If the controller* (4) *is defined by* (39) *and* $\mathscr{V}_N(\xi(0))$ *is non-empty, then the evolution of the closed-loop system* (5) *is such that* $\mathscr{V}_N(\xi(k))$ *is non-empty and*

$$\lim_{k\to\infty} v_0^*(\xi(k)) = 0\,. \tag{42}$$

for all $d(\cdot) \in \mathscr{M}_{\mathscr{D}}$ *and all* $k \in \mathbb{N}$.

Proof. See Appendix A.5. □

We can now state our second main result:

Theorem 2 (Offset removal and constraint satisfaction). *Suppose that Assumptions 1–4 hold and that the controller* (4) *is defined by* (39)*. One can choose the initial controller state* $\sigma(0)$ *such that* $\mathbb{P}_N(\xi(0))$ *has a solution and the evolution of the closed-loop system* (5) *satisfies* (9) *for all* $d(\cdot) \in \mathscr{M}_{\mathscr{D}}$ *and all* $k \in \mathbb{N}$ *if and only if the initial plant state* $x(0) \in X_N^{\mathbf{v}}$.

Proof. See Appendix A.6. □

As in Section 3.3, we need to initialize the controller state correctly such that $\mathbb{P}_N(\xi(0))$ has a solution. A sensible method for simultaneously obtaining an optimal initial controller state and input perturbation sequence is to solve the following, given the initial plant state $x(0)$:

$$\left(\hat{x}(0), \hat{d}(0), \mathbf{v}^*(\xi(0))\right) := \arg\min_{(\hat{x},\hat{d},\mathbf{v})} \left\{ J_N(\mathbf{v}) + \lambda\left((\hat{x}-x)^T(\hat{x}-x) + \hat{d}^T\hat{d}\right) \,|\mathbf{v} \in \mathscr{V}_N(\xi) \text{ and } x = x(0) \right\}, \tag{43}$$

where λ is a strictly positive scalar.

4.2 Efficient Implementation of the Receding Horizon Controller

Recall that $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$ are polyhedral sets given by a finite number of affine inequality constraints. As a consequence, it is easy to obtain an equivalent expression for the set of admissible input perturbations $\mathscr{V}_N(\xi)$ as

$$\mathscr{V}_N(\xi) = \left\{\mathbf{v} \in \mathbb{R}^{mN} \,\middle|\, F\mathbf{v} \le b + G\mathbf{d} + H\xi \text{ for all } \mathbf{d} \in \mathscr{D}^N\right\}, \tag{44}$$

where the matrices $F \in \mathbb{R}^{q\times mN}$, $G \in \mathbb{R}^{q\times rN}$, $H \in \mathbb{R}^{q\times 3n}$ and the vector $b \in \mathbb{R}^q$ depend on the augmented system dynamics (32) and are given in Appendix B.

The following result, which is a restatement of [17, Prop. 1], allows one to efficiently compute an equivalent expression for $\mathscr{V}_N(\xi)$ in terms of a finite number of affine inequality constraints:

Proposition 2 (Expression for $\mathscr{V}_N(\xi)$). *If* $\mathscr{V}_N(\xi)$ *is given as in* (44)*, then*

$$\mathscr{V}_N(\xi) = \left\{\mathbf{v} \in \mathbb{R}^{mN} \mid F\mathbf{v} \le c + H\xi \right\}, \tag{45a}$$

where

$$c := b + \operatorname{vec}\min_{\mathbf{d}\in\mathscr{D}^N} G\mathbf{d} \tag{45b}$$

and $\operatorname{vec}\min_{\mathbf{d}\in\mathscr{D}^N} G\mathbf{d} := [\min_{\mathbf{d}\in\mathscr{D}^N} G_1\mathbf{d} \ \cdots \ \min_{\mathbf{d}\in\mathscr{D}^N} G_q\mathbf{d}]^T$*;* G_i *denotes the i'th row of G.*

Remark 6. Since $\mathscr{D}$ (and hence $\mathscr{D}^N$) is a polyhedron and can therefore be described by a finite number of affine inequality constraints, c can be computed efficiently by solving q LPs.

Remark 7. If $\mathscr{D}$ is given only by upper and lower bounds on the components of d, then it is not necessary to solve LPs in order to compute c; checking the signs of the components of G is sufficient [17]. For example, if the disturbance is assumed to take on values in the hypercube

$$\mathscr{D} := \{d \in \mathbb{R}^r \mid \|d\|_\infty \leq \eta\},$$

then it is easy to show (c.f. [17, Prop. 2]) that

$$c = b - \eta \operatorname{abs}(G)\mathbf{1}_{rN},$$

where the components of the matrix $\operatorname{abs}(G)$ are the absolute values of the corresponding components of G.

Remark 8. From Appendix B it is clear that the number of constraints q in (45a) is not dependent on the description for $\mathscr{D}$, but only dependent on N and the number of constraints that describe $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$. Note also that q increases only linearly with the horizon length N.

Since one can obtain a polyhedral expression for $\mathscr{V}_N(\xi)$, it is possible to compute a polyhedral expression for $X_N^{\mathbf{v}}$, defined in (40), by using standard projection algorithms [13, 14], i.e.

$$X_N^{\mathbf{v}} = \operatorname{Proj}_X \left\{(\xi, \mathbf{v}) \in \mathbb{R}^{3n} \times \mathbb{R}^{mN} \mid F\mathbf{v} \leq c + H\xi\right\}.$$

Given all of the above, it is now clear that the minimizer to $\mathbb{P}_N(\xi)$ exists if and only if $\mathscr{V}_N(\xi) \neq \emptyset$ and that the minimizer is the solution to the following finite-dimensional strictly convex quadratic program (QP):

$$\mathbf{v}^*(\xi) = \arg\min_{\mathbf{v}} \{J_N(\mathbf{v}) \mid F\mathbf{v} \leq c + H\xi\}. \tag{46}$$

There are essentially two ways in which one can compute $v_0^*(\xi)$ (and hence the control input) for a given ξ:

- As is standard in conventional model predictive control [6, 7, 15], given the current value for ξ, one can compute $v_0^*(\xi)$ on-line by solving the QP defined in (46) using standard QP solution methods.
- The QP in (46) is a so-called *parametric* QP, since the constraints (and hence the solution) of the QP in (46) are dependent on the *parameter* ξ. This observation allows one to compute the explicit expression for $v_0^*(\cdot)$ off-line using recent results presented in [18]. The results in [18] can be used to show that $v_0^*(\cdot)$ is a piecewise affine function of ξ and is defined over a polyhedral partition, i.e. the domain of $v_0^*(\cdot)$ is the union of a finite number of polyhedra and $v_0^*(\cdot)$ is affine in each polyhedron. Computing $v_0^*(\xi)$ on-line amounts to looking up the polyhedron that contains the current value of ξ and substituting ξ into the corresponding affine function.

We conclude this section by pointing out that, because of the above, (43) is also a finite-dimensional strictly convex QP.

5 Illustrative example

As an example, we consider a jacketed continuous stirred tank reactor (CSTR) studied by Henson and Seborg [19] in which an irreversible liquid-phase reaction occurs. A detailed nonlinear model has two states (reactant concentration and reactor temperature), one input (cooling liquid temperature) and two disturbances (feed temperature and feed reactant concentration). This CSTR shows three steady states, two of which are open-loop unstable, and for quality and safety reasons the middle conversion open-loop

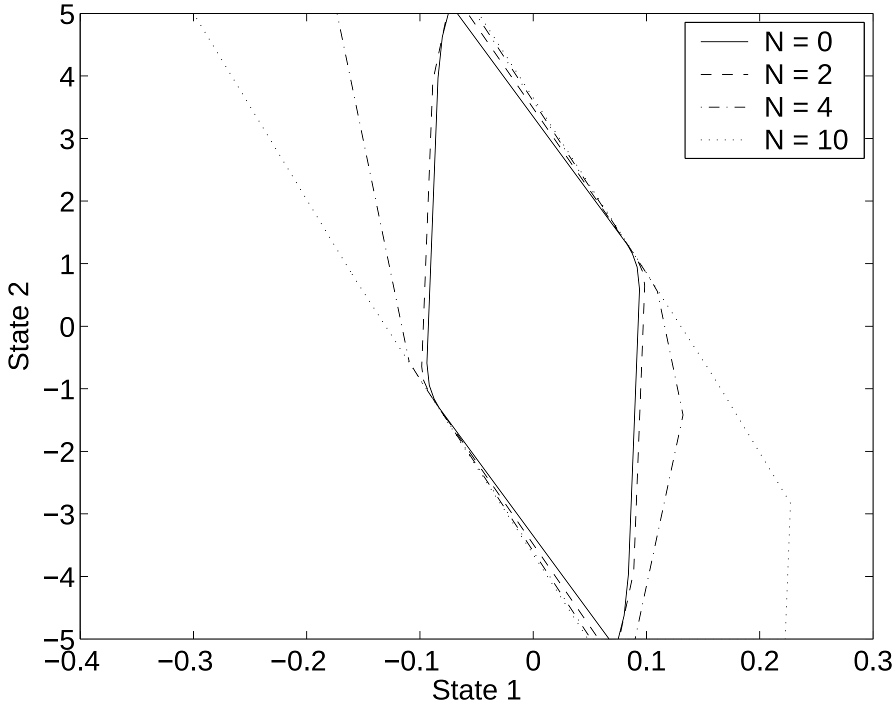

Figure 1: Domain of ($X_N^{\mathbf{v}}$) for different fixed horizons

unstable steady-state is chosen as a desired operating setpoint. Using a sampling time of $t_s = 0.1$ min and introducing deviation variables (from the corresponding steady state) a linearized model is as follows:

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}^+ = \begin{bmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix} u + \begin{bmatrix} -0.0002 & 0.0893 \\ 0.1390 & 1.2267 \end{bmatrix} \begin{bmatrix} d^1 \\ d^2 \end{bmatrix}$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},$$

in which x^1 and x^2 represent the reactant concentration and the reactor temperature, respectively; u represents the coolant temperature; d^1 and d^2 represent the feed temperature and the feed reactant concentration, respectively. Notice from the structure of C_z that the controlled variable is the reactor temperature, for which offset-free control to the origin is required. Also notice that the system matrix A has one stable and one unstable eigenvalue. The following constraints are considered:

$$\begin{bmatrix} -0.5 \\ -5 \end{bmatrix} \leq \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \leq \begin{bmatrix} 0.5 \\ 5 \end{bmatrix}, \quad -15 \leq u \leq 15, \quad \begin{bmatrix} -5 \\ -0.1 \end{bmatrix} \leq \begin{bmatrix} d^1 \\ d^2 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 0.1 \end{bmatrix}.$$

We present in Figure 1 the domain of attraction (i.e. $X_N^{\mathbf{v}}$) of the proposed controller obtained with different fixed horizons (specified in the figure), using the same stabilizing gain K computed as the optimal LQR gain with $Q = C_z^T C_z$ and $R = 0.1$ as penalty matrices. As expected from Theorem 1 we have that an increase in the fixed horizon length results in a larger domain of attraction. Note that, since the number of inputs is equal to the number of controlled variables ($m = p = 1$), the steady-state target is uniquely defined by (14).

We present in Figure 2 the domain of attraction of the linear controller (i.e. X_0) obtained with different stabilizing gain matrices. These gains were computed as the optimal LQR gain with $Q = C_z^T C_z$ and different R (specified in the figure) as penalty matrices. It is interesting to notice that when the input penalty matrix

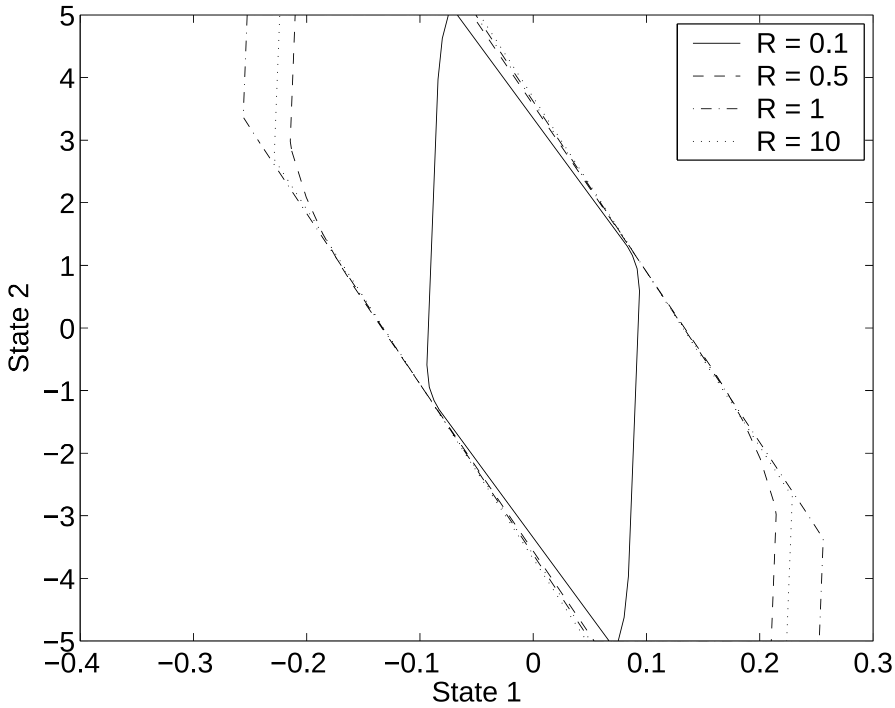

Figure 2: Domain of attraction (X_0) for different stabilizing gain

R used to compute the stabilizing gain is increased a larger domain of attraction set is usually obtained. However, when $R = 10$ the domain of attraction is smaller than that obtained with $R = 1$.

We present in Figure 3 the closed-loop simulation results (controlled variable and input, respectively) obtained with the proposed receding horizon controller based on three different stabilizing gain matrices. These gain matrices were obtained as the optimal LQR gain with $Q = C_z^T C_z$ and different R (specified in the figure). The fixed horizon used is $N = 5$ for all controllers, the penalty matrix used in (37a) is $W = 1$ and the scalar used in (43) was $\lambda = 1000$. The initial plant state is $x(0) = \begin{bmatrix} -0.1 & 2 \end{bmatrix}^T$, and there is no disturbance in the time interval $[0,4]$ minutes. Then, the disturbance is $d = \begin{bmatrix} 5 & 0 \end{bmatrix}^T$ in the time interval $[4,8]$ minutes. Next, the disturbance is $d = \begin{bmatrix} 5 & 0.1 \end{bmatrix}^T$ in the time interval $[8,12]$ minutes. Finally, the disturbance is $d = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T$ in the time interval $[12,16]$ minutes. As expected the proposed controllers asymptotically drive the controlled variable to the origin despite the presence of persistent unmeasured disturbances. Moreover, it is interesting to notice that the choice of the stabilizing gain has a direct impact on the closed-loop performance. That is, when the stabilizing gain is computed using lower input penalty R, the disturbance is rejected more quickly and a larger control input is used.

6 Conclusions

This paper has shown how one can design a nonlinear, time-invariant, dynamic state feedback controller that guarantees constraint satisfaction and offset-free control in the presence of a persistent, non-stationary, additive disturbance on the state. The design of the controller was split into two parts:

- The design of a dynamic, linear, time-invariant controller. A deadbeat observer is used to estimate

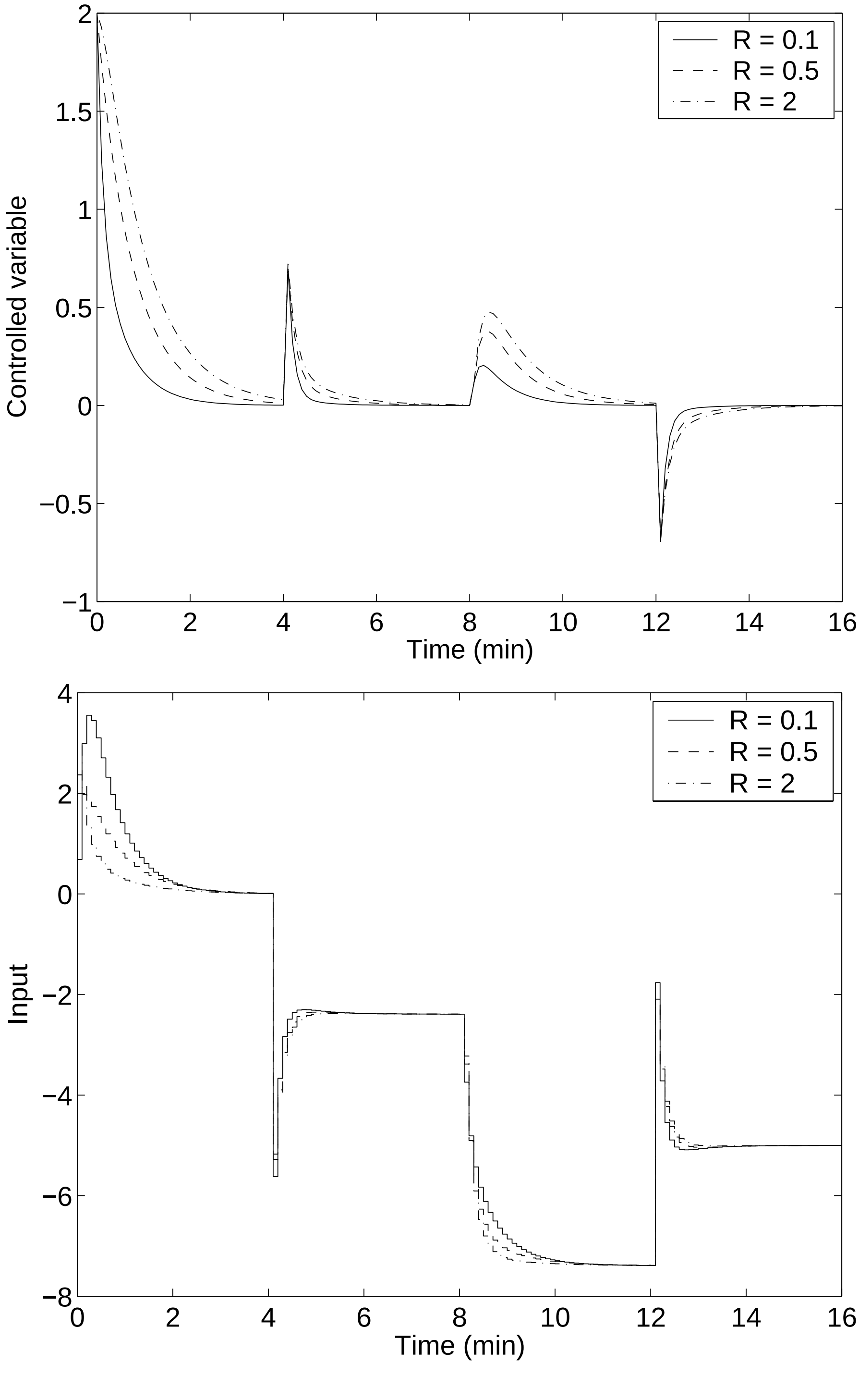

Figure 3: Closed-loop simulation results for different receding horizon controllers: controlled variable (top) and input (bottom).

the disturbance, the new steady-state is given as a linear function of the current plant and observer states and the controller aims to regulate the plant state and input to the new target steady-state. In order to estimate the region of attraction of the linear controller, it was proposed that the maximal constraint-admissible robustly positively invariant set $\mathcal{O}_\infty$ associated with the linear controller be computed.

- The design of a dynamic, nonlinear, time-invariant receding horizon controller. In order to increase the region of attraction of the linear controller, a robust receding horizon controller, which computes perturbations to the linear control law, was proposed. The receding horizon controller includes the state and input constraints explicitly in its computations as well as the effect of the unknown persistent disturbance, thereby guaranteeing robust constraint satisfaction. It was proposed that the set $\mathcal{O}_\infty$ be included as a terminal constraint in the prediction horizon and it was shown that the specific formulation of the proposed receding horizon controller improves on the linear controller in terms of the domain of attraction.

The robust receding horizon controller presented in this paper can be implemented in an efficient manner and is computationally tractable. The incorporation of the effect of the disturbance has very little effect on the computational complexity since the number of decision variables and constraints increases only linearly with an increase in the horizon length.

The paper also demonstrated the effectiveness of using the results in this paper in designing a controller for guaranteeing offset-free control of a continuous stirred tank reactor. The simulation results were shown to be in agreement with the theory.

7 Acknowledgements

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Appendices

A Proofs

A.1 Proof of Lemma 1

The statement follows immediately from the KKT conditions for (15) [20, Sect. 16.1]. It is important to verify that the matrix to be inverted in (17) is non-singular.

In order to see this, let Z be a matrix of dimension $(n+m)\times(m-p)$ (if the system is square, i.e. $m = p$, the proof of non-singularity is trivial) whose columns are an orthonormal basis for the null space of $\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix}$. Consider any vector $v \in \mathbb{R}^{m-p}$ with $v \neq 0$, and let

$$z = \begin{bmatrix} x^* \\ u^* \end{bmatrix} = Zv\,.$$

Notice that since the columns of Z are independent, $z \neq 0$.

We now show by contradiction that $u^* \neq 0$. Suppose that $u^* = 0$. We can write

$$\begin{bmatrix} I-A \\ C_z \end{bmatrix} x^* = \begin{bmatrix} Bu^* \\ 0 \end{bmatrix} = 0\,.$$

From Assumption 1 we have that (A, C_z) is detectable, which implies from the Hautus Lemma [21, Sect. 7.1] that the matrix $\begin{bmatrix} I-A \\ C_z \end{bmatrix}$ has full column rank. However, this implies that $x^* = 0$, which is in contradiction with the fact that $z \neq 0$. Hence, it must be that $u^* \neq 0$.

Therefore, since $z = Zv$, we can write:

$$v^T Z^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{R} \end{bmatrix} Zv = \begin{bmatrix} x^* \\ u^* \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{R} \end{bmatrix} \begin{bmatrix} x^* \\ u^* \end{bmatrix} = (u^*)^T \bar{R} u^* > 0\,,$$

where the last inequality comes from the fact that $\bar{R}$ is positive definite and that $u^* \neq 0$. This implies that the reduced Hessian defined as

$$Z^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{R} \end{bmatrix} Z$$

is positive definite, and we can apply the results in [20, Lemma 16.1] to deduce that

$$\begin{bmatrix} 0 & 0 & -I+A^T & -C_z^T \\ 0 & \bar{R} & B^T & 0 \\ I-A & -B & 0 & 0 \\ C_z & 0 & 0 & 0 \end{bmatrix}$$

is non-singular and that the target calculation (15) has a unique minimizer.

A.2 Proof of Lemma 2

From the definitions, it follows that

$$\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K} = \begin{bmatrix} A+BK+B\Gamma & -B\Gamma & B\Gamma \\ I_n + A + BK + B\Gamma & -I_n - B\Gamma & I_n + B\Gamma \\ I_n & -I_n & I_n \end{bmatrix}. \tag{47}$$

The eigenvalues of $\mathscr{A}+\mathscr{B}\mathscr{K}$ are the roots of $\det(\mathscr{A}+\mathscr{B}\mathscr{K}-\lambda I_{3n})=0$. Note that

$$\begin{aligned}
\det(\mathscr{A}+\mathscr{B}\mathscr{K}-\lambda I_{3n}) &= \det\left(\begin{bmatrix} A+BK+B\Gamma-\lambda I_n & -B\Gamma & B\Gamma \\ I_n+A+BK+B\Gamma & -I_n-B\Gamma-\lambda I_n & I_n+B\Gamma \\ I_n & -I_n & I_n-\lambda I_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} A+BK+B\Gamma-\lambda I_n & -B\Gamma & B\Gamma \\ \lambda I_n & -\lambda I_n & \lambda I_n \\ I_n & -I_n & I_n-\lambda I_n \end{bmatrix}\right) \text{ (subtract rows 1 and 3 from 2)} \\
&= \det\left(\begin{bmatrix} A+BK-\lambda I_n & -B\Gamma & B\Gamma \\ 0 & -\lambda I_n & \lambda I_n \\ 0 & -I_n & I_n-\lambda I_n \end{bmatrix}\right) \text{ (add column 2 to column 1)} \\
&= \det\left(\begin{bmatrix} A+BK-\lambda I_n & 0 & B\Gamma \\ 0 & 0 & \lambda I_n \\ 0 & -\lambda I_n & I_n-\lambda I_n \end{bmatrix}\right) \text{ (add column 3 to column 2)} \\
&= (-1)^n\cdot\det\left(\begin{bmatrix} A+BK-\lambda I_n & 0 & B\Gamma \\ 0 & -\lambda I_n & I_n-\lambda I_n \\ 0 & 0 & \lambda I_n \end{bmatrix}\right) \text{ (exchange rows 2 and 3)} \\
&= (-1)^n\cdot\det(A+BK-\lambda I_n)\cdot\det(-\lambda I_n)\cdot\det(\lambda I_n) \text{ (block triangular matrix)} \\
&= (-1)^n\cdot\det(A+BK-\lambda I_n)\cdot(-\lambda)^n\cdot\lambda^n \\
&= (-1)^{2n}\cdot\lambda^{2n}\cdot\det(A+BK-\lambda I_n)\,.
\end{aligned}$$

This implies that $2n$ of the eigenvalues of $\mathscr{A}+\mathscr{B}\mathscr{K}$ are at the origin and the rest are equal to the eigenvalues of $A+BK$. Hence, if $A+BK$ has all its eigenvalues strictly inside the unit disk, then the eigenvalues of $\mathscr{A}+\mathscr{B}\mathscr{K}$ are strictly inside the unit disk.

A.3 Proof of Lemma 3

Since $\lim_{k\to\infty} d(k)=\bar{d}$ we have from (22)–(23) and from the results of Lemma 2 that

$$\lim_{k\to\infty}\xi(k)=\xi_\infty=\mathscr{A}_{\mathscr{K}}\xi_\infty+\mathscr{E}\bar{d}=\mathscr{A}\xi_\infty+\mathscr{B}u_\infty+\mathscr{E}\bar{d}, \tag{48}$$

in which $u_\infty := \mathscr{K}\xi_\infty$. Let ξ_∞ be partitioned as follows:

$$\xi_\infty=\begin{bmatrix} x_\infty \\ \hat{x}_\infty \\ \hat{d}_\infty \end{bmatrix},$$

in which each block is a column vector of length n. We can rewrite (48) explicitely as follows:

$$x_\infty = Ax_\infty+Bu_\infty+E\bar{d} \tag{49a}$$
$$\hat{x}_\infty = Ax_\infty+Bu_\infty+(x_\infty-\hat{x}_\infty+\hat{d}_\infty) \tag{49b}$$
$$\hat{d}_\infty = x_\infty-\hat{x}_\infty+\hat{d}_\infty\,. \tag{49c}$$

From (49c) we immediately obtain that:

$$x_\infty=\hat{x}_\infty\,,$$

which, combined with (49b), leads to

$$x_\infty = Ax_\infty+Bu_\infty+(x_\infty-\hat{x}_\infty+\hat{d}_\infty)\,. \tag{50}$$

Let $(\bar{x}_\infty, \bar{u}_\infty)$ denote the solution to the target calculation problem (15) for the augmented state ξ_∞. From (15b) we can write:

$$\bar{x}_\infty = A\bar{x}_\infty + B\bar{u}_\infty + (x_\infty - \hat{x}_\infty + \hat{d}_\infty)\,, \tag{51}$$

which, subtracted from (50), leads to:

$$x_\infty - \bar{x}_\infty = A(x_\infty - \bar{x}_\infty) + B(u_\infty - \bar{u}_\infty) = (A+BK)(x_\infty - \bar{x}_\infty)\,, \tag{52}$$

where the last step comes from (18). It is important to notice that (52) and Assumption 3 implies that

$$x_\infty = \bar{x}_\infty\,. \tag{53}$$

In order to see this, note that (52) can be rewritten as $(I_n - A - BK)(x_\infty - \bar{x}_\infty) = 0$, which is certainly satisfied if (53) holds. It is also clear that (53) is the unique solution if $(I_n - A - BK)$ is full rank. Suppose that $(I_n - A - BK)$ is not full rank and let $x^* \in \mathbb{R}^n$ be such that $x^* \neq 0$ and $(I_n - A - BK)x^* = 0$. We would have $x^* = (A+BK)x^*$, that is x^* is an eigenvector of $(A+BK)$ associated with the eigenvalue $\lambda^* = 1$, which is in contradiction with Assumption 3 because all eigenvalues of $(A+BK)$ are strictly inside the unit circle. Hence, $(I_n - A - BK)$ is full rank and (53) holds. Finally, from (53) and from (15b) we obtain:

$$\begin{aligned} 0 &= C_z\bar{x}_\infty = C_z x_\infty \\ &= \begin{bmatrix} C_z & 0 \end{bmatrix} \xi_\infty \\ &= \lim_{k\to\infty} \begin{bmatrix} C_z & 0 \end{bmatrix} \xi(k)\,. \end{aligned}$$

A.4 Proof of Theorem 1

It follows trivially from Assumption 4 that X_0 contains the origin in its interior. The rest of the proof is by induction.

Let the plant state $x \in X_i^{\mathbf{v}}$, where $i \in \{1,\dots,N-1\}$, the controller state σ be such that $\mathscr{V}_i(\xi)$ is non-empty and $\mathbf{v}_i := (v_0,\dots,v_{i-1}) \in \mathscr{V}_i(\xi)$ be an admissible perturbation sequence of length i. Also let $\mathbf{d}_i := (d_0,\dots,d_{i-1}) \in \mathscr{D}^i$ be an admissible disturbance sequence of length i.

From the definition of $\mathscr{V}_i(\xi)$, it follows that $\chi(i,\xi,\mathbf{v}_i,\mathbf{d}_i) \in \mathscr{O}_\infty$ for all $\mathbf{d}_i \in \mathscr{D}^i$. Recall that $\mathscr{O}_\infty$ is disturbance invariant and constraint-admissible for the closed-loop system (23), hence $\mathscr{O}_\infty$ is disturbance invariant and constraint-admissible for system (32) under the infinite perturbation sequence $\{v(k)\}_{k=0}^\infty := \{0,0,\dots\}$.

It follows that if $\chi(i,\xi,\mathbf{v}_i,\mathbf{d}_i) \in \mathscr{O}_\infty$ for all $\mathbf{d}_i \in \mathscr{D}^i$, then $\chi(i+1,\xi,(\mathbf{v}_i,0),\mathbf{d}_{i+1}) \in \mathscr{O}_\infty$ for all $\mathbf{d}_{i+1} \in \mathscr{D}^{i+1}$. This implies that if $\mathbf{v}_i \in \mathscr{V}_i(\xi)$, then $(\mathbf{v}_i,0) \in \mathscr{V}_{i+1}(\xi)$. Hence if $\mathscr{V}_i(\xi)$ is non-empty, then $\mathscr{V}_{i+1}(\xi)$ is non-empty. It follows from the definition of $X_i^{\mathbf{v}}$ that if $x \in X_i^{\mathbf{v}}$, then $x \in X_{i+1}^{\mathbf{v}}$, hence $X_i^{\mathbf{v}} \subseteq X_{i+1}^{\mathbf{v}}$.

Using similar arguments as above, the result is completed by noticing that $X_0 \subseteq X_1^{\mathbf{v}}$.

A.5 Proof of Lemma 4

The method of proof is standard.

Assume $\mathscr{V}_N(\xi)$ is non-empty and let $\mathbf{v}^*(\xi) := \left(v_0^*(\xi),\dots,v_{N-1}^*(\xi)\right)$ be the associated minimizer of problem $\mathbb{P}_N(\xi)$. Consider also the candidate perturbation sequence for the augmented state ξ^+ at the next time instant, i.e.

$$\tilde{\mathbf{v}}(\xi) := \left(v_1^*(\xi),\dots,v_{N-1}^*(\xi),0\right)\,.$$

Using similar arguments as in the proof of Theorem 1, given the set of possible augmented states $f(\xi,\mathscr{D})$ at the next time instant, it follows that if $\xi^+ \in f(\xi,\mathscr{D})$, then $\tilde{\mathbf{v}}(\xi)$ is an admissible input perturbation sequence (satisfying input, state and terminal constraints for all allowable disturbances), i.e. $\tilde{\mathbf{v}}(\xi) \in \mathscr{V}_N(f(\xi,d))$ for all $d \in \mathscr{D}$. This proves that if $\mathscr{V}_N(\xi(0))$ is non-empty, then $\mathscr{V}_N(\xi(k))$ is non-empty for all $k \in \{1,2,\ldots\}$ and all allowable disturbance sequences.

If we let $J_N^*(\xi) := J_N(\mathbf{v}^*(\xi))$, then it follows that $J_N^*(\xi) = J_N(\mathbf{v}^*(\xi)) \geq J_N(\tilde{\mathbf{v}}(\xi)) \geq J_N(\mathbf{v}^*(\xi^+)) = J_N^*(\xi^+)$ for all $\xi^+ \in f(\xi,\mathscr{D})$. This implies that, for all allowable disturbances, the sequence $\{J_N(\mathbf{v}^*(\xi(k))\}_{k=0}^{\infty}$ is a non-negative, non-increasing sequence. Hence, it converges to some non-negative value, which implies that

$$\lim_{k\to\infty} J_N(\mathbf{v}^*(\xi(k))) - J_N(\mathbf{v}^*(\xi(k+1))) = 0\,.$$

However, we can write (recalling that W is positive definite)

$$\begin{aligned} 0 &\leq v_0^*(\xi(k))^T W v_0^*(\xi(k)) = J_N(\mathbf{v}^*(\xi(k))) - J_N(\tilde{\mathbf{v}}(\xi(k))) \\ &\leq J_N(\mathbf{v}^*(\xi(k))) - J_N(\mathbf{v}^*(\xi(k+1)))\,, \end{aligned}$$

which implies that

$$\lim_{k\to\infty} v_0^*(\xi(k))^T W v_0^*(\xi(k)) = 0\,,$$

and also that

$$\lim_{k\to\infty} v_0^*(\xi(k)) = 0\,,$$

where we used the fact that W is positive definite.

A.6 Proof of Theorem 2

Sufficiency. Suppose that $x(0) \in X_N^{\mathbf{v}}$, then it immediately follows from (40) that one can choose a controller state $\sigma(0)$ such that $\mathscr{V}_N(\xi(0)) \neq \emptyset$ and hence $\mathbb{P}_N(\xi(0))$ has a solution. This implies from Lemma 4 we have that $\mathscr{V}_N(\xi(k)) \neq \emptyset$ for all $k \in \mathbb{N}$ and also that

$$v_\infty := \lim_{k\to\infty} v(k) := \lim_{k\to\infty} v_0^*(\xi(k)) = 0\,. \tag{54}$$

The fact that (9a) holds can now be shown exactly as in the proof of Lemma 3, since from (32) and (54) it follows that

$$\begin{aligned} \lim_{k\to\infty} \xi(k) = \xi_\infty &= \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{B}v_\infty + \mathscr{E}\bar{d} \\ &= \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{E}\bar{d} \\ &= \mathscr{A}\xi_\infty + \mathscr{B}u_\infty + \mathscr{E}\bar{d}\,, \end{aligned}$$

in which $u_\infty = \mathscr{K}\xi_\infty + v_\infty = \mathscr{K}\xi_\infty$.

The fact that (9b) holds follows trivially from Lemma 4 and the definition of $\mathscr{V}_N(\cdot)$.

Necessity. This is obvious because if $x(0) \notin X_N^{\mathbf{v}}$, then we either have that $x(0) \notin \mathscr{X}$ or that for all $\sigma(0) \in \mathbb{R}^{2n}$, $\mathscr{V}_N(\xi(0)) = \emptyset$ and hence the control input is undefined at time 0.

B Computation of Matrices in Section 4.2

Let the polyhedra $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$ be defined by

$$\mathscr{X} := \{x \in \mathbb{R}^n \mid S_x x \leq b_x\}\,, \tag{55}$$

$$\mathscr{U} := \{u \in \mathbb{R}^m \mid S_u u \leq b_u\}\,, \tag{56}$$

$$\mathscr{O}_\infty := \left\{\xi \in \mathbb{R}^{3n} \;\middle|\; S_\xi \xi \leq b_\xi\right\}\,, \tag{57}$$

where $S_x \in \mathbb{R}^{q_x \times n}$, $S_u \in \mathbb{R}^{q_u \times m}$, $S_\xi \in \mathbb{R}^{q_\xi \times 3n}$, $b_x \in \mathbb{R}^{q_x}$, $b_u \in \mathbb{R}^{q_u}$, $b_\xi \in \mathbb{R}^{q_\xi}$ and let the matrices $T_x \in \mathbb{R}^{q_x \times 3n}$ and $T_u \in \mathbb{R}^{q_u \times 3n}$ be defined as

$$T_x := \begin{bmatrix} S_x & 0 \end{bmatrix}, \quad T_u := S_u \mathscr{K}. \tag{58}$$

Given the above, it follows from (36) that

$$\mathscr{V}_N(\xi) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; \begin{array}{c} \xi_0 = \xi,\ T_x \xi_k \le b_x,\ k = 1, \dots, N-1,\ S_\xi \xi_N \le b_\xi \text{ and} \\ T_u \xi_k + S_u v_k \le b_u,\ k = 0, \dots, N-1 \text{ for all } \mathbf{d} \in \mathscr{D}^N \end{array} \right\}. \tag{59}$$

Let

$$q := (N-1) q_x + N q_u + q_\xi \tag{60}$$

and the matrices $L \in \mathbb{R}^{q \times mN}$ and $M \in \mathbb{R}^{q \times (N+1)3n}$ be given by

$$L := \begin{bmatrix} 0 \\ I_N \otimes S_u \end{bmatrix}, \tag{61}$$

$$M := \begin{bmatrix} 0 & I_{N-1} \otimes T_x & 0 \\ 0 & 0 & S_\xi \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_N \otimes T_u & 0 \end{bmatrix}. \tag{62}$$

If we let the block vectors $b \in \mathbb{R}^q$ and $\mathbf{x} \in \mathbb{R}^{3n(N+1)}$ be defined as

$$b := \begin{bmatrix} \mathbf{1}_{N-1} \otimes b_x \\ b_\xi \\ \mathbf{1}_N \otimes b_u \end{bmatrix}, \quad \mathbf{x} := \begin{bmatrix} \xi_0 \\ \vdots \\ \xi_N \end{bmatrix}, \tag{63}$$

then it is easy to verify from (59) that

$$\mathscr{V}_N(\xi) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; \xi_0 = \xi,\ L\mathbf{v} + M\mathbf{x} \le b \text{ for all } \mathbf{d} \in \mathscr{D}^N \right\}. \tag{64}$$

If we now let the block matrices $\mathbf{A} \in \mathbb{R}^{3n(N+1) \times 3n}$, $\mathbf{B} \in \mathbb{R}^{3n(N+1) \times mN}$ and $\mathbf{E} \in \mathbb{R}^{3n(N+1) \times rN}$ be defined as

$$\mathbf{A} = \begin{bmatrix} I \\ \mathscr{A}_\mathscr{K} \\ \mathscr{A}_\mathscr{K}^2 \\ \vdots \\ \mathscr{A}_\mathscr{K}^N \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mathscr{B} & 0 & \dots & 0 \\ \mathscr{A}_\mathscr{K}\mathscr{B} & \mathscr{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_\mathscr{K}^{N-1}\mathscr{B} & \mathscr{A}_\mathscr{K}^{N-2}\mathscr{B} & \dots & \mathscr{B} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mathscr{E} & 0 & \dots & 0 \\ \mathscr{A}_\mathscr{K}\mathscr{E} & \mathscr{E} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_\mathscr{K}^{N-1}\mathscr{E} & \mathscr{A}_\mathscr{K}^{N-2}\mathscr{E} & \dots & \mathscr{E} \end{bmatrix}, \tag{65}$$

then it follows that

$$\mathbf{x} = \mathbf{A}\xi_0 + \mathbf{B}\mathbf{v} + \mathbf{E}\mathbf{d}. \tag{66}$$

Finally, by substituting (66) into (64) it follows that

$$\mathscr{V}_N(\xi) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; F\mathbf{v} \le b + G\mathbf{d} + H\xi \text{ for all } \mathbf{d} \in \mathscr{D}^N \right\}, \tag{67}$$

where

$$F := L + M\mathbf{B}, \quad G := -M\mathbf{E}, \quad H := -M\mathbf{A}. \tag{68}$$

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