On the stability of end-to-end congestion control for the Internet

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Abstract

In [2] Johari and Tan consider an Internet type network with distributed congestion control of the form proposed by Kelly *et al* in [3], and determine a sufficient condition for local stability of the network under the condition that all round trip times are equal. They conjecture that the same condition will also guarantee local stability when the round trip times are disparate. The continuous time version of this conjecture is true.

Notation

$\sigma(Z)$ denotes the spectrum of a square matrix Z and $\rho(Z)$ its spectral radius. Sets are always indexed by i, and $\{f_i\}$ is used as an abbreviation for $\{f_i : i = 1, 2, \ldots\}$. In particular, $\mathrm{Co}\{x_i\}$ denotes the convex hull of the set of points $\{x_1, x_2, \ldots\}$ and $\mathrm{diag}\{x_i\}$ denotes the matrix with the elements $x_1, x_2, \ldots$ on the leading diagonal and zeros elsewhere.

1 Introduction

In this section we review the material in [2], derive the necessary transfer function matrices and the give the main stability result by appealing to Theorem 1. This theorem is actually stated and proven in Section 2. Section 2 is self-contained, in an attempt to make clear which properties of the network are being exploited in the proof.

Johari and Tan consider a network where the rate of marked packets received back at the source of route r is

$$z_r(t) = x_r(t - T_r) \sum_{j \in r} \mu_j\big(t - d_2(j, r)\big) \tag{1}$$

where the summation is taken over all resources used by route r and

$$\mu_j(t) = p_j\left(\sum_{q:j \in q} x_q\big(t - d_1(j, q)\big)\right) \tag{2}$$

denotes the marking rate at the resource j. Here, the summation is taken over all routes which are using this resource. It is assumed that

$$d_1(j, r) + d_2(j, r) = T_r \; \forall r$$

where $d_1(j, r)$ denotes the forward delay from the source of route r to the resource j and $d_2(j, r)$ denotes the return delay via the recipient. T_r thus denotes the total round trip delay on each route. The function $p_j(\cdot)$ is assumed to be continuous, differentiable and non-decreasing. The sending rate on each route is then regulated according to the rule

$$\dot{x}_r = k_r(w_r - z_r). \tag{3}$$

If we write $x_r(t) = \hat{x}_r + y_r(t)$, then these equations may be linearized about the equilibrium $z_r = w_r = \hat{x}_r \sum_{j \in r} p_j$ to give

$$z_r(t) = \hat{x}_r \sum_{j \in r} p_j + \sum_{j \in r} p_j y_r(t - T_r) + \sum_{j \in r} \sum_{q:j \in q} \hat{x}_r p'_j y_q\big(t - d_1(j, q) - d_2(j, r)\big)$$

neglecting higher order terms, where p_j represents $p_j\left(\sum_{q:j \in q} \hat{x}_q\right)$ and p'_j the derivative evaluated at this point.

Taking Laplace transforms, we obtain $\bar{z}(s) = G(s)\bar{y}(s)$ where $\bar{y}_r(s) = \mathcal{L}y_r(t)$ and $\bar{z}_r(s) = \mathcal{L}(z_r - \hat{x}_r \sum_{j \in r} p_j)$. The transfer function $G(s)$ has the form

$$G(s) = \operatorname{diag}\{\exp(-sT_i)\}\big(XM(s) + WX^{-1}\big)$$

where

$$M_{rq}(s) = \sum_{j \in q \cap r} p'_j \exp\Big(-s\big(d_1(j, q) - d_1(j, r)\big)\Big),$$

$X = \mathrm{diag}\{\hat{x}_i\}$ and $W = \mathrm{diag}\left\{\hat{x}_i \sum_{j\in i} p_j\right\}$. Let $P(s) = M(s) + X^{-2}W$, so $G(s) = \mathrm{diag}\{\exp(-sT_i)\}XP(s)$ where $P(s) = P^T(-s)$ and $P(j\omega) > 0 \forall\omega$. [1]

We also define $\bar{w}_r(s) = \mathcal{L}(w_r - \hat{x}_r \sum_{j\in r} p_j)$, to give

$$\bar{y}_r(s) = \frac{k_r}{s}\big(\bar{w}_r(s) - \bar{z}_r(s)\big).$$

As noted in [2], $\|e_r^T P(j\omega)X\|_1$, the rth absolute row sum of the matrix $P(j\omega)X$, can be bounded as

$$\|e_r^T P(j\omega)X\|_1 \le \sum_{j\in r} p_j + \sum_{j\in r} p'_j \sum_{q:j\in q} \hat{x}_q \quad \forall\omega.$$

Theorem 1 below thus shows that the network described by (1)–(3) is locally stable if

$$k_r \left(\sum_{j\in r} p_j + \sum_{j\in r} p'_j \sum_{q:j\in q} \hat{x}_q\right) < \frac{\pi}{2T_r} \forall r.$$

This is the continuous time version of the conjecture in [2].

2 Main result

Consider a delay system[2] with transfer function

$$G(s) = \mathrm{diag}\big\{\exp(-sT_i)\big\}XP(s) \tag{4}$$

where $P(s) = P^T(-s)$, $P(j\omega) > 0\forall\omega$, $X = \mathrm{diag}\{x_i\}$, $x_i \in \mathbb{R}_+$ and $T_i \in \mathbb{R}_+$. A controller is given by

$$K(s) = \mathrm{diag}\left\{\frac{k_i}{s}\right\} \tag{5}$$

where $k_i \in \mathbb{R}_+$, and these are connected as

$$\bar{z}(s) = G(s)\bar{y}(s), \quad \bar{y}(s) = K(s)\big(\bar{w}(s) - \bar{z}(s)\big). \tag{6}$$

The following theorem gives a sufficient condition for stability of this interconnection.

[1] This elegant decomposition into the product of a diagonal and a Hermitian matrix is crucial to the solution of the problem and is taken directly from [2].

[2] by which we mean $\mathcal{L}^{-1}G(s) = \sum_i Z_i\delta(t-\tau_i)$, $\tau_i \ge 0,\ Z_i \in \mathbb{R}^{n\times n}\ \forall i$

Theorem 1. *The closed loop system described by* (4), (5) *and* (6) *is asymptotically stable if*

$$k_i \|e_i^T P(j\omega) X\|_1 < \frac{\pi}{2T_i} \; \forall i, \omega.$$

Proof. Fix $\omega \in \mathbb{R}_+$ and put

$$Q = \frac{2}{\pi} \operatorname{diag}\big\{\sqrt{k_i T_i x_i}\big\} P(j\omega) \operatorname{diag}\big\{\sqrt{k_i T_i x_i}\big\}, \quad L = \operatorname{diag}\left\{\frac{\pi}{2} \frac{\exp(-j\omega T_i)}{j\omega T_i}\right\}.$$

By assumption,

$$\rho(Q) = \rho\left(\operatorname{diag}\left\{\frac{2T_i k_i}{\pi}\right\} P X\right) < 1$$

(since the spectral radius of any matrix is bounded by its maximum absolute row sum) and so Lemma 1 states that

$$\sigma\big(K(j\omega)G(j\omega)\big) = \sigma(QL) \subset \rho(Q) \operatorname{Co}\left(0 \cup \left\{\frac{\pi}{2} \frac{\exp(-j\omega T_i)}{j\omega T_i}\right\}\right).$$

Now, the convex hull of all points $\frac{\pi}{2}\frac{\exp(-jx)}{jx}$, $x \in \mathbb{R}_+$, includes the point -1 on its boundary (at $x = \pi/2$) and also includes the origin (see Fig. 1). Since $\rho(Q) < 1$, it follows that

$$-1 \notin \rho(Q) \operatorname{Co}\left(0 \cup \left\{\frac{\pi}{2} \frac{\exp(-j\omega T_i)}{j\omega T_i}\right\}\right).$$

and hence that the eigenloci of $K(j\omega)G(j\omega)$ cannot enclose the point -1. Consequently, by the generalized Nyquist criterion ([1]), the closed loop system is asymptotically stable. □

Lemma 1. *Let* $Q = Q^* > 0$ *and* $L = \operatorname{diag}\{l_i\}$, $l_i \in \mathbb{C}$, $\forall i$ *be given. Then*

$$\sigma(QL) \subset \rho(Q) \operatorname{Co}(0 \cup \{l_i\}).$$

Proof. Let v be a normalized eigenvector of QL, corresponding to an eigenvalue λ then

$$QLv = \lambda v$$

and so

$$Lv = \lambda Q^{-1} v \implies \lambda = \frac{v^* L v}{v^* Q^{-1} v} = \rho(Q) \left(\sum_i \frac{|v_i|^2}{k} l_i + (1 - \frac{1}{k}) \cdot 0\right)$$

where $k = \rho(Q)(v^* Q^{-1} v) \geq 1$. □

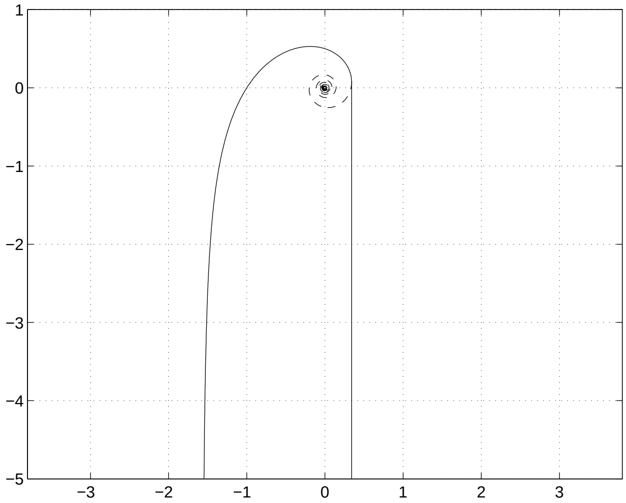

Figure 1: $\frac{\pi}{2}\exp(-jx)/(jx)$ and its convex hull

3 Notes

A slightly weaker form of the conjecture, with 1 replacing $\pi/2$ has recently been shown in [4]. In the present framework, this corresponds to bounding the eigenloci to the right of a vertical line through -1, which requires a reduction in gain by a factor $2/\pi$.

In the original discrete time conjecture, the controller has the form

$$x_r(t+1) = x_r(t) + k_r(w_r - z_r)$$

and local stability is conjectured to be guaranteed provided $k_r < 2\sin\left(\frac{\pi}{2(2T_r+1)}\right)$. Taking z-transforms and proceeding as in the proof of Theorem 1 would lead to conclusion that the eigenloci lie in the convex hull of the *family* of curves

$$2\sin\left(\frac{\pi}{2(2T_r+1)}\right)\frac{\exp(-j\theta T_r)}{\exp(j\theta)-1}, \quad \theta \in [0,\pi]$$

which now vary with T_r. Whilst -1 lies on the boundary of each of these curves, and they each enjoy the appropriate local convexity, the convex hull in fact contains the point -1 in its interior. This does not necessarily mean that a counterexample exists, but does mean that any proof would need to be more sophisticated. Calculations indicate that the convex hull cuts the negative real axis at around -1.0005, so a very slightly weaker version of the discrete time conjecture is certainly true.

I would like to thank Frank Kelly for bringing this problem to my attention.

References

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