Robust feasibility in model predictive control: Necessary and sufficient conditions

Eric C. Kerrigan and Jan M. Maciejowski

Control Group, Department of Engineering, University of Cambridge
Trumpington Street, Cambridge CB2 1PZ, United Kingdom
Tel: +44-1223-339222/2732, Fax: +44-1223-332662
eck21@eng.cam.ac.uk,jmm@eng.cam.ac.uk
http://www-control.eng.cam.ac.uk/

Abstract

A number of results are derived for analysing the robust feasibility of a given Model Predictive Control (MPC) scheme which ignores model mismatch and/or disturbances during control input computation. The main contribution of this paper is the development of computationally tractable tests for determining the robust feasibility of an MPC controller for linear or piecewise affine systems, where the constraints are given by the union of convex polyhedra and the disturbance acts additively on the state. Practical tests are also presented which allow one to give robust feasibility guarantees for all optimal and sub-optimal MPC control actions.

Keywords: nonlinear model predictive control, constraints, robust feasibility, robust strong feasibility, robust set invariance theory, piecewise affine systems.

1 Introduction

Over the last few decades Model Predictive Control (MPC) has proven to be a very successful technique for the control of multivariable, constrained systems. However, most industrially-implemented MPC schemes do not explicitly take into account that there is a mismatch between the actual plant and the model used in the synthesis of the controller. In practice, it often happens that the MPC controller or a disturbance drives the system to a state which is outside the so-called *feasible set* of the associated finite horizon optimal control problem. As a result, a control input cannot be computed and the optimisation problem has to be redefined, e.g. by softening the state constraints [8].

However, in order to guarantee robust constraint satisfaction in safety-critical applications it is desirable that infeasibility of the MPC optimisation problem is avoided at all costs. In other words, once inside the feasible set the system evolution should remain inside the feasible set for all time and for all disturbance sequences. In this paper, if an MPC controller is such that the closed-loop system has this property, it will be called *robustly feasible*. Results are presented which allow one to determine *a priori* whether or not a given MPC controller which ignores disturbances in the optimal control problem, is robustly feasible.

The main concepts discussed here apply to general, nonlinear systems and are directly related to well-established results in set invariance theory [4, 5]. One of the contributions of this paper is explicitly linking some of these results with the robustness analysis of MPC controllers. Another contribution of this paper is the development of computationally tractable robust feasibility tests for linear, time-invariant (LTI) and piecewise affine (PWA) systems. Theorems 8 and 9 consider the case when the optimal solution is found at each time step and Section 4 develops results which allow one to test whether or not robust feasibility is guaranteed for all optimal and sub-optimal solutions to the MPC problem.

Most previous robust invariance tests have been concerned with LTI systems with LTI control laws, e.g. [7]. In this paper test for checking robust invariance (which is equivalent to robust feasibility) are developed for LTI and PWA systems, where the control law is a *single-valued PWA map* [1, 2] or a *non-convex, set-valued map*. This paper is also similar in emphasis to that of [10, 11] and the references contained in the latter, in the sense that robustness analysis, rather than robust synthesis, of MPC controllers is the main topic of the paper. However, whereas [10, 11] contain sufficient conditions only, this paper derives necessary and sufficient conditions for guaranteeing robust feasibility.

Notation: In definitions $:=$ reads "is defined as" and $=:$ reads "defines". $\operatorname{Arg\,min}_{\theta\in\Theta} f(\theta)$ is the set of all minimisers of $\min_{\theta\in\Theta} f(\theta)$. If $f : A \to B$, then $f(A) := \{f(x) \in B : x \in A\}$ and hence $f(A) = \bigcup_{x\in A} f(x)$. The Minkowski (vector) sum $A \oplus B := \{x + y : x \in A, y \in B\}$. Given a set A, the interior is denoted by A°, the complement by A^c and the set of all subsets of A (power set) is denoted by 2^A. The set inclusion $A \subset B$ holds iff $A \subseteq B$ and $A \neq B$. If $f(\cdot) := [f_1(\cdot), f_2(\cdot), \ldots]'$ is a vector function and $v := [v_1, v_2, \ldots]'$ is a vector, then $v \geq \max_{\theta\in\Theta} f(\theta)$ iff $\forall i : v_i \geq \max_{\theta\in\Theta} f_i(\theta)$. The unit vector $\vec{1} := [1, 1, \ldots, 1]'$.

2 Problem Description

The discrete-time plant dynamics are given by

$$x_{k+1} = f_{\mathrm{p}}(x_k, u_k, w_k)$$

where $k \in \mathbb{Z}$ is the time instant, $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the control input and $w_k \in \mathbb{R}^d$ is an unknown disturbance. At each time instant the disturbance is randomly selected from a closed and bounded set $\mathbf{W} \subset \mathbb{R}^d$.

Due to physical, safety and/or performance constraints, the design requirement is that both the computed control input and the state be constrained to closed and bounded[1] sets $\mathbf{U} \subset \mathbb{R}^m$ and $\mathbf{X} \subset \mathbb{R}^n$, respectively.

For the design of the MPC controller, a nominal, discrete-time model

$$x_{k+1} = f_{\mathrm{m}}(x_k, u_k)$$

is used and the disturbance is *ignored*. It is assumed that both $f_{\mathrm{p}} : \mathbf{X} \times \mathbf{U} \times \mathbf{W} \to \mathbb{R}^n$ and $f_{\mathrm{m}} : \mathbf{X} \times \mathbf{U} \to \mathbb{R}^n$ are single-valued maps.

This paper is concerned with deriving results and tests which allow one to determine whether or not the resulting state of the plant in closed-loop with an MPC controller, where the effect of the disturbance has been neglected during controller design, robustly satisfies the constraint $\mathbf{X}$.

At each time instant the current state x is measured and the MPC control action $\mu_N(x) := u_0^*(x)$ is computed, where $u_0^*(x)$ is the first element of a solution to the following open-loop optimal control problem:

Problem 1 (Finite Horizon Optimal Control). *Find a*

$$\mathbf{u}^*(x) \in \operatorname{Arg}\min_{\mathbf{u}} F(x_P) + \sum_{k=0}^{P-1} \ell(x_k, u_k)$$

subject to

$$x_{k+1} = f_{\mathrm{m}}(x_k, u_k),\, x_k \in \mathbf{X},\, u_k \in \mathbf{U},\, k = 0, \dots, P-1 \quad \text{(1a)}$$
$$u_k = h(x_k),\, k = N, \dots, P-1 \quad \text{(1b)}$$
$$x_0 = x,\, x_P \in \mathbf{T}, \quad \text{(1c)}$$

where

$$\mathbf{u} := (u_0, \dots, u_{N-1}), \quad \mathbf{u}^*(x) := \left(u_0^*(x), \dots, u_{N-1}^*(x)\right).$$

In the above, N is the control horizon and P is the prediction horizon with $P \geq N \geq 1$. As is common in MPC, a terminal constraint[2] $\mathbf{T} \subseteq \mathbf{X}$ and terminal control law[3] $h : \mathbf{X} \to \mathbb{R}^m$ may also be used in defining the control problem. It is assumed that the terminal cost $F : \mathbf{X} \to \mathbb{R}$ and stage cost $\ell : \mathbf{X} \times \mathbf{U} \to \mathbb{R}$ attain their minima inside their respective domains.

As can be seen, the existence of a solution to the above problem is dependent on the current state x. For a given x, the constraints (1) define the set of all *feasible input sequences*

$$\mathbf{C}_N(x) := \left\{\mathbf{u} \in \mathbf{U}^N : (x, \mathbf{u}) \,\text{satisfies}\,(1)\right\}.$$

The *feasible set* $\mathcal{K}_N$ of the above optimisation problem[4] is the set of states for which the resulting constraints define a non-empty $\mathbf{C}_N(x)$, i.e.

$$\mathcal{K}_N := \{x \in \mathbb{R}^n : \exists \mathbf{u} \in \mathbf{C}_N(x)\} = \{x \in \mathbb{R}^n : \mathbf{C}_N(x) \neq \emptyset\}.$$

If $x \in \mathcal{K}_N$, then a solution exists to the optimisation problem and hence the MPC control input is defined for the given state. For all $x \notin \mathcal{K}_N$, a control input cannot be computed.

Remark 2. If f_{m} and h are linear maps and $\mathbf{X}, \mathbf{U}, \mathbf{W}$ and $\mathbf{T}$ are compact, convex polyhedra, then $\mathcal{K}_N$ is also a compact, convex polyhedron and can be computed via projection or Minkowski summation [5, Chap. 5]. If f_{m} and h are piecewise affine (PWA) maps, then $\mathcal{K}_N$ can still be computed. In this case, $\mathcal{K}_N$ is not necessarily convex but can still be described by the union of convex polyhedra [5, Chap. 4].

Because of the presence of both state and input constraints[5], the feasible set is not necessarily equal to either of $\mathbb{R}^n$ or $\mathbf{X}$, but is only a subset of $\mathbf{X}$, i.e. $\mathcal{K}_N \subseteq \mathbf{X}$; in practice the feasible set is often a strictly proper subset of $\mathbf{X}$, i.e. $\mathcal{K}_N \subset \mathbf{X}$.

3 Optimal Solutions

Note that though the assumptions in Section 2 guarantee the existence of a solution to the finite horizon optimal control problem for all $x \in \mathcal{K}_N$, the solution is not necessarily unique. This section is concerned with robust feasibility when the solution to the finite horizon optimal control problem is unique. For the case of guaranteeing robust feasibility when the optimal solution is not unique, the development follows similar lines of reasoning as in Section 4 and will not be discussed here.

Assumption 3 (Uniqueness). For all $x \in \mathcal{K}_N$, a unique solution to Problem 1 exists.

The above assumption holds only in this section and is made in order to guarantee that the MPC controller $\mu_N : \mathcal{K}_N \to \mathbf{U}$ is a single-valued map.

Definition 4 (Robust feasibility). The MPC controller is robustly feasible if and only if for all states inside the feasible set and for all disturbances inside $\mathbf{W}$, the state of the plant at the next time instant lies inside the feasible set, i.e. μ_N robustly feasible $\Leftrightarrow \forall x \in \mathcal{K}_N : f_{\mathrm{p}}(x, \mu_N(x), \mathbf{W}) \subseteq \mathcal{K}_N$.

Guaranteeing robust feasibility is very strongly linked with ideas in set invariance [5]. The concept of the *reach set* of the closed-loop system is particularly useful in this context. For the closed-loop system, the reach set from an arbitrary set of states $\Omega \subset \mathbb{R}^n$ is defined as

$$\mathcal{R}_{\mu_N}(\Omega) := \left\{f_{\mathrm{p}}(x, \mu_N(x), w) \in \mathbb{R}^n : x \in \Omega, w \in \mathbf{W}\right\}$$

and it follows that $\mathcal{R}_{\mu_N}(\Omega) = \bigcup_{x \in \Omega} f_{\mathrm{p}}(x, \mu_N(x), \mathbf{W})$.

Given this, the following result follows trivially from the definition of a robustly feasible MPC controller:

[1] In order to improve the numerical robustness of the controller, in practice it is a good idea to add upper and lower bounds for each state, even though the original design requirements may not translate into a bounded $\mathbf{X}$.

[2] Note that for the results in this paper to hold, an invariance condition on the terminal constraint does not have to be satisfied.

[3] If $N = P$, then a terminal control law is not included in the problem and hence (1b) is removed. Note also that a stabilising condition on the terminal control law or upper bound by F on the cost-to-go are not necessary for the results in this paper to hold.

[4] If $\mathcal{K}_N$ is the feasible set of the MPC problem with a control horizon of N and a prediction horizon of P, then $\mathcal{K}_{N-1}$ is the feasible set of a problem with a control horizon of $N-1$ and a prediction horizon of $P-1$.

[5] If there are no state or terminal constraints ($\mathbf{X} = \mathbf{T} = \mathbb{R}^n$) then $\mathcal{K}_N = \mathbb{R}^n$ and hence feasibility is never a problem and a control input can always be computed.

Proposition 5 (Unique solution). *The MPC controller μ_N is robustly feasible if and only if $\mathcal{R}_{\mu_N}(\mathcal{K}_N) \subseteq \mathcal{K}_N$.*

Remark 6. This result is not surprising since, by definition, the MPC controller is robustly feasible if and only if $\mathcal{K}_N$ is a robustly positively invariant set for the closed-loop system[6]. It is well-known that a given set is robustly positively invariant if and only if the set of states reachable from the given set is contained within itself [4, Sect. 3.2].

Remark 7. It is important to note that in order for the MPC controller to be robustly feasible it is necessary that $\mathcal{K}_N$ is control invariant[7]; if the feasible set is not control invariant, then $\mathcal{R}_{\mu_N}(\mathcal{K}_N) \nsubseteq \mathcal{K}_N$. However, it should be stressed that, depending on the model dynamics and choice of parameters in (1), $\mathcal{K}_N$ might already be control invariant without requiring any modification; it is recommended that the designer check for control invariance before modifying the MPC controller. A control invariant feasible set can be guaranteed in a number of ways, e.g. by setting the control and prediction horizons sufficiently large or by choosing a control invariant $\mathbf{T}$ [5, 6].

In order to derive a test or algorithm which implements the above condition some structure regarding the problem has to be assumed. An explicit expression for the MPC control law also needs to be derived. Fortunately this is possible for some classes of systems.

Recent results have shown how to compute an explicit expression of the receding horizon control law, which is implicitly defined by the MPC control problem, for LTI or PWA with polyhedral constraints [1, 2]. In both these papers, with the choice of an appropriate cost function, the resulting control law was shown to be PWA, i.e.

$$\mu_N(x) =: K_i x + g_i, \text{ for } x \in \mathcal{X}_i, \quad i = 1, \ldots, s, \tag{2}$$

where each (K_i, g_i) is such that $K_i \in \mathbb{R}^{m\times n}$, $g_i \in \mathbb{R}^m$. All the $\mathcal{X}_i$ are closed[8], convex polyhedra such that their interiors $\mathcal{X}_i^\circ$ are pairwise disjoint, i.e. $i \neq j \Leftrightarrow \mathcal{X}_i^\circ \cap \mathcal{X}_j^\circ = \emptyset$, and the union of all $\mathcal{X}_i$ is equal to the feasible set

$$\mathcal{K}_N = \bigcup_{i\in\mathcal{I}} \mathcal{X}_i, \quad \mathcal{I} := \{1, 2, \ldots, s\}.$$

Theorem 8 (Robust feasibility test: LTI). *If $f_\mathrm{p}(x,u,w) = f_\mathrm{m}(x,u) + Ew = Ax + Bu + Ew$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, E \in \mathbb{R}^{n\times d}$, the MPC control law is given by* (2) *and the feasible set*

$$\mathcal{K}_N =: \left\{x \in \mathbb{R}^n : Gx \leq v, G \in \mathbb{R}^{q\times n}, v \in \mathbb{R}^q\right\}$$

has been computed, then the MPC controller is robustly feasible if and only if

$$v \geq \max_{i,x,w} \left\{G\left((A + BK_i)x + Bg_i + Ew\right) : w \in \mathbf{W}, x \in \mathcal{X}_i, i \in \mathcal{I}\right\}.$$

[6] Ω is robustly positively invariant $\Leftrightarrow \forall x \in \Omega : f_\mathrm{p}(x, \mu_N(x), \mathbf{W}) \subseteq \Omega$.

[7] Ω is control invariant $\Leftrightarrow \forall x \in \Omega$, $\exists u \in \mathbf{U}$ such that $f_\mathrm{m}(x,u) \in \Omega$.

[8] Strictly speaking, each $\mathcal{X}_i$ is not guaranteed to be closed, but a closed approximation of each $\mathcal{X}_i$ is used when computing the explicit control law.

It can be seen that if $\mathbf{W}$ is a closed, convex polyhedron, then the above test can be implemented by solving $q \times s$ linear programs, where q is the number of linear inequalities describing the feasible set and s is the number of convex polyhedral partitions in the MPC control law.

If both the plant and model are PWA, then a similar test can also be derived, since an MPC control law can be computed as described in [1] such that the closed-loop system is PWA. Let the plant dynamics be PWA, i.e. $f_\mathrm{p}(x,u,w) = A_i x + B_i u + E_i w + c_i$ for $(x,u) \in \mathcal{C}_i$ and the MPC control law also be PWA. The closed-loop system is now given by the PWA dynamics

$$f_\mathrm{p}(x, \mu_N(x), w) = \Phi_i x + E_i w + b_i, \text{ for } x \in \mathcal{X}_i, \ i = 1, \ldots, s, \tag{3}$$

where each $\Phi_i \in \mathbb{R}^{n\times n}, E_i \in \mathbb{R}^{n\times d}, b_i \in \mathbb{R}^n$ and all the $\mathcal{X}_i$ are closed, convex polyhedra. The feasible set $\mathcal{K}_N = \bigcup_{i\in\mathcal{I}} \mathcal{X}_i$ can be computed as described in [5, Chaps 4–5], is not necessarily convex, but is given by the union of the convex polyhedra $\mathcal{X}_i$.

Because of the non-convexity of the feasible set, in order to implement a robust feasibility test, it is easiest if the complement of the feasible set $\mathcal{K}_N^\mathrm{c}$ is computed. Once $\mathcal{K}_N$ is available, computing the complement is straightforward [5, App. D]. The complement is given by the union of t open, convex polyhedra $\mathcal{Q}_j$, i.e.

$$\mathcal{K}_N^\mathrm{c} = \bigcup_{j\in\mathcal{J}} \mathcal{Q}_j, \quad \mathcal{J} := \{1, 2, \ldots, t\}.$$

Theorem 9 (Robust feasibility test: PWA). *If the closed-loop system is given by* (3) *and the complement of the feasible set $\mathcal{K}_N^\mathrm{c} = \bigcup_{j\in\mathcal{J}} \mathcal{Q}_j$ has been computed, where each*

$$\mathcal{Q}_j =: \left\{x \in \mathbb{R}^n : H_j x < z_j, H_j \in \mathbb{R}^{q_j\times n}, z_j \in \mathbb{R}^{q_j}\right\},$$

then the MPC controller is robustly feasible if and only if for all $(i,j) \in \mathcal{I} \times \mathcal{J}$,

$$0 < \min_{\varepsilon,x,w} \left\{\varepsilon : H_j\left(\Phi_i x + E_i w + b_i\right) < z_j + \vec{1}\varepsilon, w \in \mathbf{W}, x \in \mathcal{X}_i\right\}.$$

As before, if $\mathbf{W}$ is a closed, convex polyhedron then the above test can be implemented by solving a finite number of linear programs.

Once the receding horizon control law, the feasible set and its complement have been computed, robust feasibility tests for LTI and PWA systems can also be derived using techniques based on those described in [3]. However, it is felt that perhaps Theorems 8 and 9 are easier to understand and implement, since the results presented here need not be implemented by setting up and solving *mixed-integer* linear programs, as is required for the (more general) problem formulated in [3], but only require solving standard linear programs.

4 All Optimal and Sub-optimal Solutions

In practice, especially when the system is nonlinear, one cannot guarantee that the solution is unique nor can one guarantee that the solver will return the optimal solution to Problem 1 at each time step [9, Sect. 3]. It would therefore be useful if a result could be derived which allowed one to guarantee that the MPC controller is feasible for all time and for all disturbance sequences, even if a sub-optimal control input is computed at each time step. This section is concerned with deriving such a result which guarantees robust feasibility for all optimal and sub-optimal solutions.

By ignoring the cost function in Problem 1 and treating the MPC controller as an agent which, for a given state $x \in \mathcal{K}_N$, randomly selects an admissible control input from the so-called set of *feasible inputs*, the robust feasibility problem can be addressed relatively easily on the abstract level.

Definition 10 (Feasible inputs). For a given $x \in \mathcal{K}_N$, the set of feasible inputs $\mathrm{M}_N(x)$ is the set of first elements of feasible input sequences, i.e.

$$\mathrm{M}_N(x) := \{u_0 \in \mathbb{R}^m : \mathbf{u} \in \mathbf{C}_N(x)\} .$$

Remark 11. Note that the feasible inputs $\mathrm{M}_N(x)$ differ from the set of *admissible* inputs $\mathbf{U}$; in general $\mathrm{M}_N(x) \subseteq \mathbf{U}, \forall x \in \mathcal{K}_N$. The set of feasible inputs is the subset of admissible inputs which are compatible with the constraints defining the MPC controller (1).

This definition allows one to treat the MPC controller as a *set-valued* map M_N, rather than as a single-valued map, i.e.

$$\mathrm{M}_N : \mathcal{K}_N \to 2^{\mathbf{U}} .$$

The MPC control action is now any $\mu_N(x) \in \mathrm{M}_N(x)$ and the closed-loop system is described by the *difference inclusion* $x_{k+1} \in f_{\mathrm{p}}(x_k, \mathrm{M}_N(x_k), \mathbf{W})$.

If the MPC controller is robustly feasible for all optimal and sub-optimal control inputs, it will be said to be robustly *strongly* feasible in order to distinguish this more conservative condition from the one in Section 3.

Definition 12 (Robust strong feasibility). The MPC controller is robustly strongly feasible if and only if $\forall x \in \mathcal{K}_N, w \in \mathbf{W} : f_{\mathrm{p}}(x, \mathrm{M}_N(x), w) \subseteq \mathcal{K}_N$.

For the closed-loop system, the reach set from an arbitrary set of states $\Omega \subset \mathbb{R}^n$ is defined as

$$\mathcal{R}_{\mathrm{M}_N}(\Omega) := \left\{ f_{\mathrm{p}}(x, \mathrm{M}_N(x), w) \subseteq \mathbb{R}^n : x \in \Omega, w \in \mathbf{W} \right\} .$$

It follows that $\mathcal{R}_{\mathrm{M}_N}(\Omega) = \bigcup_{x \in \Omega} f_{\mathrm{p}}(x, \mathrm{M}_N(x), \mathbf{W})$ and hence one can derive a similar feasibility result as in Section 3.

Proposition 13 (All optimal and sub-optimal solutions). *The MPC controller* M_N *is robustly strongly feasible if and only if* $\mathcal{R}_{\mathrm{M}_N}(\mathcal{K}_N) \subseteq \mathcal{K}_N$.

Again, some assumptions regarding the system structure need to be made before one can extract a practical feasibility test.

Theorem 14 (Robust strong feasibility test: LTI). *If* $f_{\mathrm{p}}(x,u,w) = f_{\mathrm{m}}(x,u) + Ew = Ax + Bu + Ew$ *and the feasible set*

$$\mathcal{K}_N =: \left\{ x \in \mathbb{R}^n : Gx \leq v, G \in \mathbb{R}^{q \times n}, v \in \mathbb{R}^q \right\}$$

has been computed, then the MPC controller is robustly strongly feasible if and only if

$$v \geq \max_{x,\mathbf{u},w} \{ G(Ax + Bu_0 + Ew) : w \in \mathbf{W}, \mathbf{u} \in \mathbf{C}_N(x) \} .$$

If $\mathbf{W}$ and (1) are given by closed, convex polyhedra, then the above test can be implemented by solving q linear programs, where q is the number of constraints describing $\mathcal{K}_N$. A similar test can be derived for piecewise affine systems, but it requires solving a number of mixed-integer linear programs.

For more complicated systems like piecewise affine systems, it might be easier to exploit the structure of the system and adopt a geometric approach. If the disturbance acts additively on the state, then the following result holds:

Theorem 15 (Additive state disturbances). *If* $f_{\mathrm{p}}(x,u,w) = f_{\mathrm{m}}(x,u) + f_{\mathrm{w}}(w)$, *then the MPC controller is robustly strongly feasible if and only if*

$$\mathcal{R}_{\mathrm{m}}(\mathcal{K}_N) \oplus f_{\mathrm{w}}(\mathbf{W}) \subseteq \mathcal{K}_N ,$$

where

$$\mathcal{R}_{\mathrm{m}}(\mathcal{K}_N) = f_{\mathrm{m}}(\mathcal{K}_N, \mathbf{U}) \cap \mathcal{K}_{N-1} .$$

Proof. See [5, Sect. 6.2]. □

It may be possible to exploit the structure when computing $f_{\mathrm{m}}(\mathcal{K}_N, \mathbf{U})$, e.g. if $f_{\mathrm{m}}(x,u) = f_{\mathrm{x}}(x) + f_{\mathrm{u}}(u)$, then $f_{\mathrm{m}}(\mathcal{K}_N, \mathbf{U}) = f_{\mathrm{x}}(\mathcal{K}_N) \oplus f_{\mathrm{u}}(\mathbf{U})$.

Corollary 16 (LTI and PWA systems).

1. *If* $f_{\mathrm{p}}(x,u,w) = f_{\mathrm{m}}(x,u) + Ew = Ax + Bu + Ew$, *then the MPC controller is robustly strongly feasible if and only if*

$$((A\mathcal{K}_N \oplus B\mathbf{U}) \cap \mathcal{K}_{N-1}) \oplus E\mathbf{W} \subseteq \mathcal{K}_N .$$

2. *If* $f_{\mathrm{p}}(x,u,w) = f_{\mathrm{m}}(x,u) + Ew = A_i x + B_i u + Ew + c_i$ *for* $x \in \mathcal{X}_i$, *then the MPC controller is robustly strongly feasible if and only if for all* $i \in \mathcal{I}$,

$$((A_i (\mathcal{X}_i \cap \mathcal{K}_N) \oplus B_i \mathbf{U} \oplus c_i) \cap \mathcal{K}_{N-1}) \oplus E\mathbf{W} \subseteq \mathcal{K}_N .$$

3. *If* $f_{\mathrm{p}}(x,u,w) = f_{\mathrm{m}}(x,u) + E_i w = A_i x + B_i u + E_i w + c_i$ *for* $x \in \mathcal{X}_i$, *then the MPC controller is robustly strongly feasible if and only if for all* $(i,j) \in \mathcal{I} \times \mathcal{I}$,

$$((A_i(\mathcal{X}_i \cap \mathcal{K}_N) \oplus B_i \mathbf{U} \oplus c_i) \cap \mathcal{X}_j \cap \mathcal{K}_{N-1}) \oplus E_j \mathbf{W} \subseteq \mathcal{K}_N .$$

Remark 17. Note that in the above corollary, it is not required that h is linear nor is it required that the constraints are compact, convex polyhedra. However, if f_p and h are linear and the constraints are given by compact, convex polyhedra, then Theorem 14 results in a simpler test.

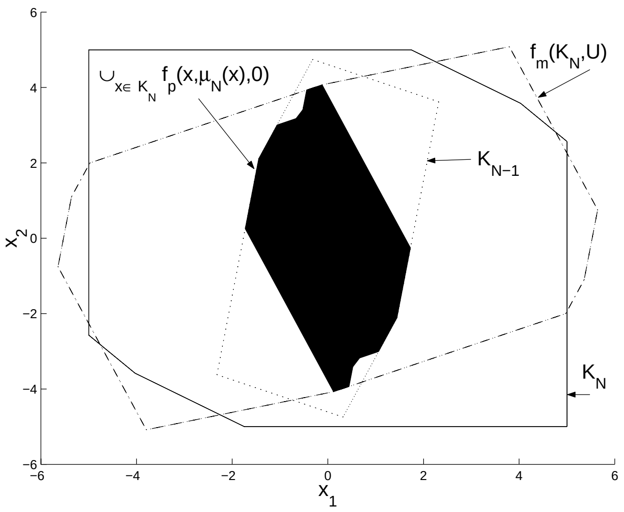

Figure 1: Plots of sets used in Example 18.

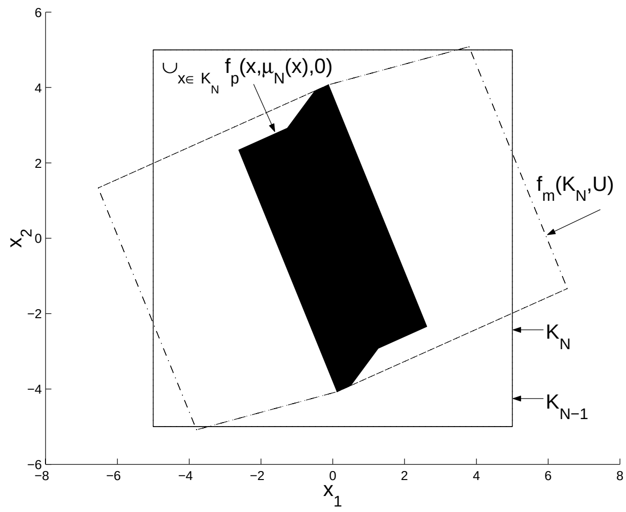

Figure 2: Plots of sets used in Example 19.

5 Examples

This section demonstrates the use of the robust feasibility and robust strong feasibility tests on simple LTI systems. The disturbance set is given by $\mathbf{W} := \{w \in \mathbb{R}^d : \|w\|_\infty \le \gamma\}$ and the aim is to determine γ_{rf} and γ_{rsf}, which are the largest values of γ for which the conditions in Theorems 8 and 14 hold, respectively. These optimised values of γ can be thought of as the robust feasibility margin and robust strong feasibility margin of the MPC controller. As is standard in robust control these values of γ can be determined iteratively by using the bisection algorithm. The explicit expressions for the MPC control laws were computed using the algorithm of [2] and the feasible sets were computed as discussed in [5].

In each example, $f_{\mathrm{p}}(x,u,w) = f_{\mathrm{m}}(x,u) + w = Ax + Bu + w$, $F(x) := 0$, $\ell(x,u) := x'x + u'u$, the input constraints $\mathbf{U}$ are given by $\|u\|_\infty \le 1$ and the state constraints $\mathbf{X}$ are given by $\|x\|_\infty \le 5$. No terminal control law is used. The feasible set $\mathcal{K}_N$ as well as $\mathcal{K}_{N-1}$ and $f_{\mathrm{m}}(\mathcal{K}_N, \mathbf{U})$ have been plotted for some of the examples. In all the figures the shaded region represents the reach set of the closed-loop system $f_{\mathrm{p}}(x, \mu_N(x), w)$ with $\mathbf{W} = \{0\}$ and is given by $\bigcup_{x\in\mathcal{K}_N} f_{\mathrm{p}}(x,\mu_N(x),0) = \bigcup_{x\in\mathcal{K}_N} f_{\mathrm{m}}(x,\mu_N(x))$.

Example 18. An open-loop stable LTI system is given by

$$x_{k+1} = \begin{bmatrix} -0.2739 & -0.6415 \\ 0.6415 & -0.2739 \end{bmatrix} x_k + \begin{bmatrix} 1.9574 \\ 0.5045 \end{bmatrix} u_k + w_k\,.$$

The horizons are $P = N = 2$ and the terminal constraint is given by $\mathbf{T} = \{x \in \mathbb{R}^2 : \|x\|_\infty \le 1\}$. After applying the results of Theorems 8 and 14, it was found that

$$\gamma_{\mathrm{rf}} = 0.9272, \quad \gamma_{\mathrm{rsf}} = 0.6317\,.$$

These results can be interpreted graphically by referring to Figure 1, Proposition 5 and Theorem 15. It can be seen that both $\bigcup_{x\in\mathcal{K}_2} f_{\mathrm{p}}(x,\mu_N(x),0)$ and $f_{\mathrm{m}}(\mathcal{K}_2,\mathbf{U}) \cap \mathcal{K}_1$ are proper subsets of the feasible set and that neither of the sets intersect the boundary of $\mathcal{K}_2$, which explains the fact that the MPC controller is both robustly feasible and robustly strongly feasible against non-zero disturbances. Also, the fact that the boundary of $\bigcup_{x\in\mathcal{K}_2} f_{\mathrm{p}}(x,\mu_N(x),0)$ is further away from the boundary of $\mathcal{K}_2$ than the boundary of $f_{\mathrm{m}}(\mathcal{K}_2,\mathbf{U}) \cap \mathcal{K}_1$ is, explains why $\gamma_{\mathrm{rf}} > \gamma_{\mathrm{rsf}}$.

Example 19. The same plant as in Example 18 is used, but with $\mathbf{T} = \mathbf{X}$. The MPC controller was found to be robustly feasible and robustly strongly feasible with

$$\gamma_{\mathrm{rf}} = 0.9272, \quad \gamma_{\mathrm{rsf}} = 0\,,$$

thereby illustrating that the robust strong feasibility condition could be considerably more conservative than the robust feasibility condition.

The reason for this large difference can be seen in Figure 2, where the feasible set $\mathcal{K}_2 = \mathcal{K}_1 = \mathbf{X}$. The set $f_{\mathrm{m}}(\mathcal{K}_2,\mathbf{U}) \cap \mathcal{K}_1$ intersects the boundary of $\mathcal{K}_2$, implying that the MPC controller is robustly strongly feasible if and only if $\mathbf{W} = \{0\}$; if the predicted state $f_{\mathrm{m}}(x,u)$ is such that it lies on the boundary of $\mathcal{K}_2$, then a non-zero disturbance exists which could drive the plant state outside the feasible set.

Example 20. The LTI system is given by

$$x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix} u_k + w_k\,.$$

The horizons are $P = N = 2$ and the terminal constraint is given by $\mathbf{T} = \{x \in \mathbb{R}^2 : \|x\|_\infty \le 1\}$.

On investigation of the relevant sets[9] it was found that $\bigcup_{x\in\mathcal{K}_2} f_{\mathrm{p}}(x,\mu_N(x),0) = f_{\mathrm{m}}(\mathcal{K}_2,\mathbf{U}) \cap \mathcal{K}_1 = \mathcal{K}_1$, $\mathcal{K}_1 \subset \mathcal{K}_2$ and that $\mathcal{K}_1$ does not intersect the boundary of $\mathcal{K}_2$. As a result, the MPC controller was found to be both robustly feasible and robustly strongly feasible for the same size of non-zero

[9] These sets are not shown due to space restrictions.

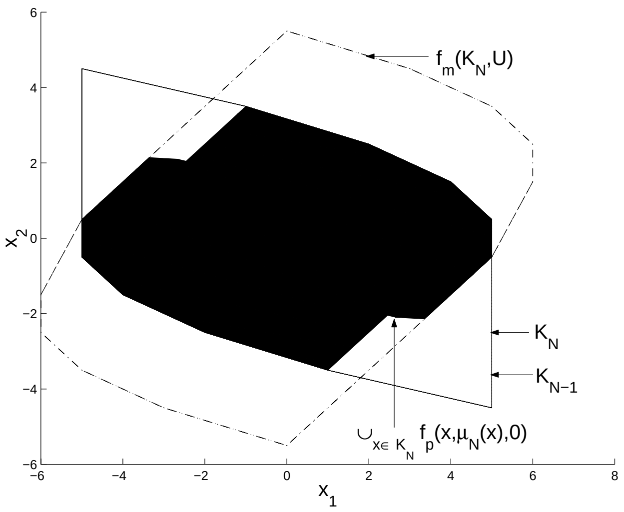

Figure 3: Plot of sets used in Example 21.

disturbance, with

$$\gamma_{\mathrm{rf}} = \gamma_{\mathrm{rsf}} = 0.333 .$$

Example 21. The LTI system is given by

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_k + w_k .$$

The horizons are $P = N = 10$ and the terminal constraint is given by $\mathbf{T} = \left\{ x \in \mathbb{R}^2 : \|x\|_\infty \leq 0.5 \right\}$. After applying the results of Theorems 8 and 14, it was found that

$$\gamma_{\mathrm{rf}} = \gamma_{\mathrm{rsf}} = 0 .$$

This result can again be interpreted graphically by referring to Figure 3, where $\mathcal{K}_{10} = \mathcal{K}_9 \subset \mathbf{X}$. In this case, $\bigcup_{x \in \mathcal{K}_{10}} f_{\mathrm{p}}(x, \mu_N(x), 0)$ intersects the boundary of the feasible set $\mathcal{K}_{10}$. The system is therefore robustly feasible (and robustly strongly feasible) if and only if $\mathbf{W} = \{0\}$. This analysis is therefore extremely useful in this case by showing that the MPC controller definitely needs to be modified in order to guarantee any kind of robust feasibility against non-zero disturbances.

6 Conclusions

This paper showed how set invariance theory could be used in understanding the robust feasibility of MPC controllers. These ideas were also applied to a more conservative feasibility condition, namely robust strong feasibility. The latter condition assures robust feasibility for all optimal and sub-optimal solutions to the MPC problem.

The test for robust feasibility requires the off-line computation of the explicit solution to the MPC problem, whereas the test for strong robust feasibility requires only the computation of the feasible set of the MPC controller. As always, there is a trade-off between computational complexity and conservativeness of the test.

Several examples showed that there exist systems for which the MPC controller has robust strong feasibility as well as systems for which the MPC controller is neither robustly feasible nor robustly strongly feasible. It still remains to be seen exactly how many combinations of plant and MPC controller are robustly feasible or robustly strongly feasible without explicitly having to take the disturbance into account during the computation of the control input.

The discussion in this paper concentrated on analysing the robust feasibility of MPC controllers which were designed without taking the uncertainty into account. Robust stability and performance have not been addressed in this paper. Further research effort could involve developing methods, based on the ideas presented in this paper, for synthesising robustly stable MPC controllers with a robust strong feasibility margin which is close or equal to the robust feasibility margin.

References

[1] A. Bemporad, F. Borrelli, and M. Morari. Optimal controllers for hybrid systems: Stability and piecewise linear explicit form. In *Proceedings of the 39th Conference on Decision and Control*, Sydney, Australia, December 2000. IEEE.

[2] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit solution of model predictive control via multiparametric quadratic programming. In *Proceedings of the 2000 American Control Conference*, Chicago, USA, June 2000. AACC/IFAC.

[3] A. Bemporad, F.D. Torrisi, and M. Morari. Performance analysis of piecewise linear systems and model predictive control systems. In *Proceedings of the 39th Conference on Decision and Control*, Sydney, Australia, December 2000. IEEE.

[4] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

[5] E.C. Kerrigan. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control*. PhD thesis, University of Cambridge, UK, November 2000. Thesis and MATLAB Invariant Set Toolbox downloadable from `http://www-control.eng.cam.ac.uk/eck21/`.

[6] E.C. Kerrigan and J.M. Maciejowski. Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control. In *Proceedings of the 39th Conference on Decision and Control*, Sydney, Australia, December 2000. IEEE.

[7] I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–367, 1998.

[8] J.M. Maciejowski. *Predictive Control with Constraints*. Prentice Hall, 2001.

[9] D.Q. Mayne. Nonlinear model predictive control: Challenges and opportunities. In F. Allgöwer and A. Zheng, editors, *Nonlinear Model Predictive Control*, volume 26 of *Progress in Systems and Control Theory*, pages 23–44. Birkhäuser Verlag, 2000.

[10] J.A. Primbs. The analysis of optimization based controllers. *Automatica*, 37:933–938, 2000.

[11] J.A. Primbs and V. Nevistić. A framework for robustness analysis of constrained finite receding horizon control. *IEEE Transactions on Automatic Control*, 45(10):1828–1837, October 2000.