ROBUSTNESS OF MULTIVARIABLE SMITH PREDICTORS

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Abstract

Stability and stability robustness of multivariable Smith predictors are analysed by viewing Smith predictors as instances of Internal Model Control. This allows a much simpler development than appears in previous analyses, and the way that delays enter the plant transfer function matrix is unrestricted. It is shown that robustness to additive plant perturbations can be predicted during design, on the basis of a model of the rational part of the plant alone. A design example based on the Shell Control Problem is included.

Keywords: Time delays, Time delay compensation, Smith predictor, Multivariable, Robustness, Internal model control, Additive perturbations.

1 Introduction

The Smith predictor is a feedback structure introduced in [14] for the control of single-input, single-output (SISO) stable processes which contain time delays. Its principal advantage is that it allows design to be performed using techniques which apply to processes with rational transfer functions. This is very important in practice because there is a wide range of algorithms and software available for use with such processes, usually described by state-space models, whereas relatively little exists for processes with irrational transfer functions, such as those containing time delays. The Smith predictor has been widely applied and analysed.

Several multivariable generalisations of the Smith predictor have been proposed. In [1] it was assumed that the delays in each input-output channel were the same. In [7] the delay in each element of a row of the plant's transfer function matrix was assumed to be the same, which is tantamount to assuming that the delays are concentrated in the output channels of the plant (each delay associated with one sensor, for example). In [2] and [11] it was assumed that the plant transfer function matrix can be factored as

$$P(s) \quad = \quad D(s)P_r(s) \qquad (1)$$

where the (i, j) element of $D(s)$ is assumed to be of the form $\exp(-sT_{ij})$ and $P_r(s)$ has only rational elements. This is less restrictive than the assumption made in [7], but still does not allow an arbitrary pattern of time delays to appear in $P(s)$. Oggunaike and Ray [10] and Palmor and Halevi [12] allow any delay to occur in each input-output channel; while this is still not completely general, it appears to be general enough for practical applications. Both of these papers propose the same multivariable generalization of the Smith predictor.

In this paper we first point out that the Hadamard (also called 'Schur'), or element-wise, matrix product can be used to describe the multivariable Smith predictor introduced in [10] and [12]. This has didactic advantages over the original presentations.

A criticism of the Smith predictor which is often made is that controllers with that structure can be very sensitive to modelling errors, particularly as regards mis-specification of time delays in the plant. This criticism is true but exaggerated. There are many control structures which are used successfully, even though the structure itself does not guarantee a good control system. For example, it is known that LQG controllers can exhibit arbitrarily bad stability margins [5], but this does not preclude the successful use of LQG controllers. It only means that stability margins of a particular design have to be checked before that design is accepted. A similar situation holds for Smith predictors (both SISO and multivariable).

As already explained above, an important and attractive feature of Smith predictors is that they can be designed without taking explicit account of the time delays in the plant. It would be very convenient to check stability margins and performance of the controller during the design, also without taking account of the time delays explicitly. In this paper we show that stability robustness against additive perturbations can indeed be checked in this way. Surprisingly, this does not appear to have been noted before for processes with general patterns of time delays, although something similar has been noted in [2] and [11] for the particular classes of processes treated in those papers. The situation is slightly worse in the multivariable than in the SISO case; for SISO plants the stability robustness to multiplicative perturbations can also be predicted in this way.

In [9] it is shown that the SISO Smith predictor is easily viewed as an instance of Internal Model Control. This makes the analysis of its stability and properties very straightforward. In this paper we follow the approach of [9], but extend it to the multivariable case. As a consequence, we also obtain a simple analysis of stability and robustness.

In particular, the robustness analysis given here is much simpler, and complementary to, the analyses given in [6], [4], [11], or [2]. The method of analysis used in [2] is similar to the one adopted here; much use is made of the Youla parameter (called the 'Q-parameter' in [2]), which is equivalent to viewing the Smith predictor as Internal Model Control, since the plant must be assumed to be stable.

2 Definition and Stability

In this section we use the Hadamard, or element-wise, matrix product, which is denoted by the operator '.*'. This product is defined as follows. If A and B are matrices of the same dimensions, then $C = A. * B$ is defined to be the matrix, of the same dimensions as A and B, whose (i, j) element c_{ij} is given by $c_{ij} = a_{ij}b_{ij}$. The Hadamard product is commutative, namely $A. * B = B. * A$, but it is not associative relative to the ordinary matrix product, namely $(A. * B)C \neq A. * (BC)$.

We assume that the (continuous-time) plant to be controlled is stable, and is described by a transfer function matrix $P(s)$. There is a time delay T_{ij} between input j and output i, and we assume that

$$P(s) \quad = \quad P_r(s).{*}D(s) \tag{2}$$

where $P_r(s)$ and $D(s)$ are defined as in (1). In other words, the time delays are the only sources of irrational elements in $P(s)$, and $P_r(s)$ is the 'rational part' of the plant. In fact $P_r(s)$ is a 'predictor', since each of its outputs is composed of predictions of signals which compose the outputs of the plant. Note that it is often very easy to obtain $P_r(s)$ and $D(s)$ for a given plant. Our assumptions on the plant correspond precisely to those made in [10] and [12].

We define the multivariable Smith predictor to be the feedback structure shown in fig.1. Here $C_r(s)$ is the transfer function matrix of any feedback controller which stabilises $P_r(s)$. The Smith predictor controller has the transfer function matrix

$$C(s) \quad = \quad C_r(s)\{I + [P_r(s) - P(s)]C_r(s)\}^{-1} \tag{3}$$

This structure clearly generalises the usual Smith predictor, and corresponds to the one introduced in [10, 12].

Figure 1 can be redrawn, as shown in fig.2, in the form of Internal Model Control. If we let

$$Q(s) \quad = \quad C_r(s)[I + P_r(s)C_r(s)]^{-1} \tag{4}$$

then it is a standard result that the feedback system shown in fig.2 is internally stable if and only if both $Q(s)$ and $P(s)$ are stable [9]. Note that the irrational nature of the various transfer functions involved does not change this standard result [3]. So long as 'stability' corresponds to analyticity in the closed right half-plane then the result holds. Since we assume that $P(s)$ is stable, internal stability of the feedback system depends only on the stability of $Q(s)$. (Note that $Q(s)$ is the Youla parameter, since $P(s)$ is stable [8].)

Since we insist that $C_r(s)$ should stabilise $P_r(s)$, $Q(s)$ is clearly stable. We have therefore established that the proposed multivariable Smith predictor is internally stable.

3 Properties

It is easy to show (and it is also a standard result [8, 9]) that the complementary sensitivity, namely the transfer function matrix from the vector of set-points r

to the vector of outputs y, is given by

$$T(s) \quad = \quad P(s)Q(s). \tag{5}$$

Substituting for $Q(s)$ from (4) we obtain

$$\begin{aligned} T(s) &= P(s)C_r(s)[I + P_r(s)C_r(s)]^{-1} && (6) \\ &= [D(s).{*}P_r(s)]C_r(s)[I + P_r(s)C_r(s)]^{-1} && (7) \end{aligned}$$

If we define $T_r(s)$ to be the complementary sensitivity which would be obtained if the feedback controller $C_r(s)$ were used with the predictor $P_r(s)$ as the 'plant', namely

$$T_r(s) \quad = \quad P_r(s)C_r(s)[I + P_r(s)C_r(s)]^{-1} \tag{8}$$

then unfortunately $T(s) \neq D(s).{*}T_r(s)$ in general.

One of the principal results obtained in the SISO case therefore does not generalise: the set-point responses are *not* the same as those obtained for the feedback combination of $P_r(s)$ with $C_r(s)$, except for the addition of the same delays as exist in the plant. However, the closed-loop poles which appear in $T(s)$ and $T_r(s)$ are the same, since both of these transfer function matrices can become unbounded only at the poles of $P_r(s)$, or the poles of $C_r(s)$, or the zeros of $[I + P_r(s)C_r(s)]$.

The prediction of performance in the face of unmeasured disturbances is not simple for the Smith predictor. The response to output disturbances is given by the sensitivity $S(s)$, and the response to input disturbances by $P(s)S(s)$, where

$$S(s) \quad = \quad I - T(s). \tag{9}$$

If we define

$$\begin{aligned} S_r(s) &= I - T_r(s) \\ &= [I + P_r(s)C_r(s)]^{-1} && (10) \end{aligned}$$

there is unfortunately no simple relationship between $S(s)$ and $S_r(s)$. But this is no worse than in the SISO case, for which the same is true [9].

Robustness in the face of unstructured multiplicative modelling errors is measured by $||T||_\infty$ [8], where $||T||_\infty = \sup_\omega \bar{\sigma}[T(j\omega)]$, where $\bar{\sigma}[.]$ denotes the maximum singular value of [.], and assuming that $T(s)$ is stable. In general $||T||_\infty \neq ||T_r||_\infty$, so robustness cannot be predicted easily in the multivariable case. This contrasts with the SISO case, for which $|T(j\omega)| = |T_r(j\omega)|$ holds.

For additive modelling errors, however, the situation is better. Robustness in the face of additive modelling errors is measured by $||CS||_\infty$ [3, 8] or, equivalently, by $||Q||_\infty$ (since $C(s) = Q(s)[I - P(s)Q(s)]^{-1}$ and $S(s) = I - P(s)Q(s)$). But for our proposed Smith predictor structure $Q(s)$ is given by (4), and hence from (10) we have

$$C(s)S(s) \quad = \quad C_r(s)S_r(s). \tag{11}$$

So the robustness to additive modelling errors is the same as the robustness of the controller $C_r(s)$ with the predictor $P_r(s)$ as the 'plant'. It can therefore be monitored easily during design, even if this is pursued using the delay-free 'plant' $P_r(s)$ only. Of course the same is true *a fortiori* for the SISO case. Surprisingly, this fact does not seem to have been noted before.

It follows immediately from the previous paragraph that the response of the control signal vector, u, to changes in the set-point vector, r, is the same for the real plant as it is for the delay-free 'plant', since both are determined by the transfer function $Q(s)$.

4 Example

As an example we take part of the 'Shell Control Problem' defined in [13]. This concerns the control of a heavy oil fractionator. Since the example is not meant to show a complete design study, we select only two outputs to be controlled — the 'Top End Point' and the 'Side End Point' — and three control inputs — the 'Top Draw', the 'Side Draw', and the 'Bottoms Reflux Duty'. (The only reasons for omitting other controlled outputs are to emphasise that the plant may have unequal numbers of inputs and outputs, and to keep the example simple.) The linearised plant model is:

$$P(s) \quad = \quad \begin{bmatrix} \frac{4.05e^{-27s}}{1+27s} & \frac{1.77e^{-28s}}{1+60s} & \frac{5.88e^{-27s}}{1+50s} \\ \\ \frac{5.39e^{-18s}}{1+50s} & \frac{5.72e^{-14s}}{1+60s} & \frac{6.90e^{-15s}}{1+40s} \end{bmatrix} \tag{12}$$

in which the time constants and delays are expressed in minutes.

The rational part of this, or the 'predictor', is:

$$P_r(s) \quad = \quad \begin{bmatrix} \frac{4.05}{1+27s} & \frac{1.77}{1+60s} & \frac{5.88}{1+50s} \\ \\ \frac{5.39}{1+50s} & \frac{5.72}{1+60s} & \frac{6.90}{1+40s} \end{bmatrix} \tag{13}$$

and the delay matrix is:

$$D(s) \quad = \quad \begin{bmatrix} e^{-27s} & e^{-28s} & e^{-27s} \\ e^{-18s} & e^{-14s} & e^{-15s} \end{bmatrix} \tag{14}$$

It is routine to construct a 5-state minimal realization of $P_r(s)$. This allows standard design algorithms, such as those based on LQG or H_∞ theory, to be used.

A design was performed on this realization, using the LQG/LTR technique (see [8], for example), which yielded a 7-state realization for $C_r(s)$:

$$\dot{z} \quad = \quad A_{Cr}z \ + \ B_{Cr}f \tag{15}$$

$$u \quad = \quad C_{Cr}z \ + \ D_{Cr}f \tag{16}$$

where the matrices $A_{Cr}, B_{Cr}, C_{Cr}, D_{Cr}$ are given in the Appendix, f is the input to $C_r(s)$, and u is the control signal. The controller has 7 states rather than 5 (the number of states needed to realize $P_r(s)$) because there are two additional states to give integral action.

For this design it was found that the singular values of the complementary sensitivities $T(s)$ and $T_r(s)$ are quite close to each other, even though there is no guarantee of this from the theory, while those of the two sensitivities $S(s)$ and $S_r(s)$ are rather different. We find that

$$\begin{aligned} ||S_r||_\infty &= 1.05, \quad ||S||_\infty = 1.51 \\ ||T_r||_\infty &= 1.02, \quad ||T||_\infty = 1.04 \end{aligned}$$

and that

$$||C_r S_r||_\infty = 0.854 = ||CS||_\infty.$$

To emphasise that good robustness to time delay variations can be obtained with Smith predictors, we performed the following exercise. Suppose that each of the time delays defined in (12) is liable to increase by up to 20% of its nominal value. Let $P_{nom}(s)$ be the plant transfer function with nominal values of the time delays, and let

$$\Delta(s) = P(s) - P_{nom}(s). \tag{17}$$

Let $\bar{\sigma}[\Delta(s)]$ denote the largest singular value of $\Delta(s)$. By generating randomly 500 plants $P(s)$ with the time delays distributed uniformly over the intervals described above, an upper bound on $\bar{\sigma}[\Delta(j\omega)]$ (as a function of ω) was generated heuristically. The reciprocal of this upper bound is shown as the upper trace in fig.3. The lower trace in the same figure shows $\bar{\sigma}[C_r(j\omega)S_r(j\omega)]$. This figure shows that

$$||C_r S_r||_\infty < \frac{1}{||\Delta||_\infty} \tag{18}$$

and hence that this design gives robust stability even in the face of up to 20% increases in the plant time delays. We point out that this design is also quite reasonable in terms of performance, with a closed-loop bandwidth (as defined by the $-3dB$ frequency of $\bar{\sigma}[T(j\omega)]$) of about 0.08 rad/min. Fig.4 shows plots of $\bar{\sigma}[T(j\omega)]$ and $\bar{\sigma}[S(j\omega)]$ against ω.

5 Conclusion

It has been shown that the stability and robustness of the multivariable Smith predictor controller which was introduced in [10] and [12] can be analysed very easily by viewing it as Internal Model Control.

In particular, robustness to additive modelling errors can be predicted during the design, using information from the delay-free model only. It is shown by example that this can be used to ensure that a particular Smith predictor design has adequate stability margins, even though such margins cannot be guaranteed in general.

The stability robustness results derived in this paper are complementary to those derived in previous analyses.

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Appendix: Realization of $C_r(s)$ for the example

$$A_{Cr} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0091 & -0.0039 & -1.4198 & -0.6702 & -0.0554 & -2.0323 & -0.0744 \\ -0.0396 & 0.0454 & 0.3358 & -0.7408 & -0.8233 & 0.5056 & -1.0476 \\ 0.0046 & -0.0022 & -0.7454 & -0.3611 & -0.0497 & -1.0956 & -0.0399 \\ 0.0222 & -0.0141 & -0.2729 & -0.8877 & -0.7224 & -0.4203 & -0.9240 \\ 0.0265 & -0.0170 & -0.3360 & -1.1109 & -0.9064 & -0.4929 & -1.1844 \end{bmatrix}$$

$$B_{Cr} = \begin{bmatrix} 0.0100 & 0 \\ 0 & 0.0100 \\ 0.0091 & -0.0039 \\ -0.0396 & 0.0454 \\ 0.0046 & -0.0022 \\ 0.0222 & -0.0141 \\ 0.0265 & -0.0170 \end{bmatrix}$$

$$C_{Cr} = \begin{bmatrix} 0 & 0 & 18.1717 & 9.1342 & 1.0337 & 26.7169 & 1.3697 \\ 0 & 0 & -5.2671 & 16.0400 & 17.3657 & -8.1907 & 22.0377 \\ 0 & 0 & 4.5720 & 23.2277 & 19.9624 & 6.7394 & 25.5366 \end{bmatrix}$$

$$D_{Cr} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6 Figures

Fig.1 The proposed multivariable Smith predictor structure.

Fig.2 The proposed controller redrawn in the form of IMC.

Fig.3 Robustness to increased time delays.
Upper trace: $1/\bar{\sigma}[\Delta(j\omega)]$
Lower trace: $\bar{\sigma}[C_r(j\omega)S_r(j\omega)]$.

Fig.4 Performance of example design.
Continuous trace: $\bar{\sigma}[S(j\omega)]$.
Dashed trace: $\bar{\sigma}[T(j\omega)]$.