Intrinsic difficulties in using the doubly-infinite time axis for input-output control theory. [1]

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Abstract. We point out that the natural definitions of stability and causality in input-output control theory lead to certain inconsistencies when inputs and outputs are allowed to have support on the doubly-infinite time-axis. In particular, linear time-invariant systems with right half plane poles cannot be considered to be both causal and stabilizable. In contrast, there is no such conflict when the semi-infinite time axis is used.
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Discussion

Consider two systems $\mathbf{P}_i$ $(i = 1, 2)$ defined in terms of convolution representations

$$y(t) = \int_{-\infty}^{\infty} h_i(t-\tau)u(\tau)d\tau = h_i * u$$

where $h_1(t) = e^t$ for $t \geq 0$ and zero otherwise, and $h_2(t) = -e^t$ for $t \leq 0$ and zero otherwise. Both systems have Laplace transfer functions $1/(s-1)$, but with differing regions of convergence. The first system is unstable and

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causal and the second is stable and non-causal (in fact anticausal) according to the usual definitions. To view these systems abstractly in the input-output setting (or in the behavioural framework of [W]) we need to work out the system trajectories for signals in some function space, say $\mathcal{L}_2$. This amounts to finding the *graph*, i.e. the set of input-output pairs in $\mathcal{L}_2$. We mention that an explicit working out of such an approach for the semi-infinite time-axis is given in [GS]. Below we consider the graphs of the two systems in the case of the doubly-infinite time-axis.

We consider first $\mathbf{P}_2$. From [R, p. 158, q. 4] $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_p$ implies $f * g \in \mathcal{L}_p$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Thus, $\mathbf{P}_2$ can be alternatively represented by the following graph on $\mathcal{L}_2(-\infty, \infty)$:

$$\hat{\mathcal{G}}_{\mathbf{P}_2,(-\infty,\infty)} = \begin{pmatrix} 1 \\ \frac{1}{j\omega - 1} \end{pmatrix} \mathcal{L}_2(-j\infty, j\infty) \tag{1}$$

after transforming to the frequency domain. (As usual, $\hat{}$ denotes the Fourier transform which maps $\mathcal{L}_2(-\infty, \infty)$ isometrically and isomorphically onto $\mathcal{L}_2(-j\infty, j\infty)$ and $\mathcal{L}_2[0, \infty)$ onto the Hardy space $\mathcal{H}_2$ of the right half plane [R, Chapter 19]. Functions in $\mathcal{H}_2$ can be extended analytically into the RHP by replacing $j\omega$ by s.)

Now consider $\mathbf{P}_1$. We will first restrict the inputs to the space $\mathcal{L}_2[0, \infty)$. Since

$$y(t) = \int_0^t e^{t-\tau} u(\tau) d\tau = e^t \int_0^t e^{-\tau} u(\tau) d\tau$$

a necessary condition for $y(t) \in \mathcal{L}_2[0, \infty)$ is that

$$\int_0^t e^{-\tau} u(\tau) d\tau \to 0$$

as $t \to \infty$, which is equivalent to $\langle e^{-t}, u \rangle = 0$. In fact, this is also sufficient since then

$$y(t) = -e^t \int_t^\infty e^{-\tau} u(\tau) d\tau = h_2 * u.$$

Thus the domain of $\mathbf{P}_1$ in $\mathcal{L}_2[0, \infty)$ is equal to the orthogonal complement of $e^{-t}\mathbb{C}$ and, moreover, $\mathbf{P}_1$ coincides with $\mathbf{P}_2$ on this domain. After taking

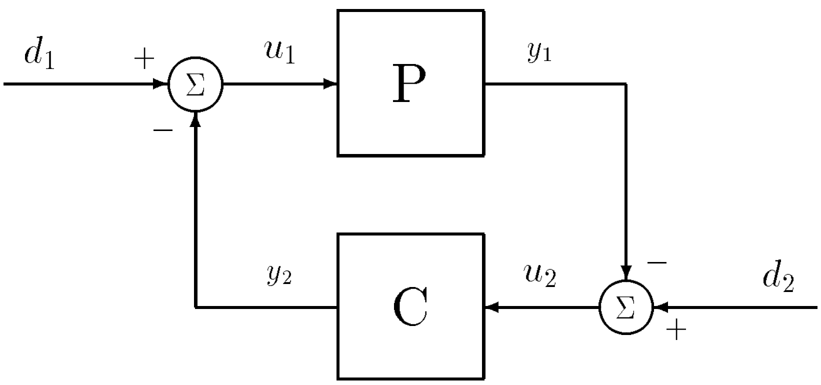

Figure 1: Standard feedback configuration

transforms, the orthogonal complement of $\frac{1}{s+1}\mathbb{C}$ is $\frac{s-1}{s+1}\mathcal{H}_2$. Therefore the graph of $\mathbf{P}_1$ restricted to signals with support on $[0,\infty)$ becomes (in the frequency domain):

$$\hat{\mathcal{G}}_{\mathbf{P}_1,[0,\infty)} = \begin{pmatrix} \frac{s-1}{s+1} \\ \frac{1}{s+1} \end{pmatrix} \mathcal{H}_2 =: G\mathcal{H}_2.$$

Since $\mathbf{P}_1$ is shift invariant, the graph of $\mathbf{P}_1$ on $\mathcal{L}_2(-\infty,\infty)$ contains (again, expressed in the frequency domain) the infinite union

$$\bigcup_{T\geq 0} e^{sT} G\mathcal{H}_2. \tag{2}$$

The graph of $\mathbf{P}_1$ on $\mathcal{L}_2(-\infty,\infty)$ is actually slightly bigger than (2) (see below) but we will not need it for the next part of the discussion.

We now turn our attention to the requirement of stabilizability in the feedback configuration of Fig. 1. Our definition of stability is the usual one: for all external $\mathcal{L}_2$ disturbance inputs, there must be solutions of the feedback equations in $\mathcal{L}_2$ so that the closed loop operators are norm bounded. If $\mathbf{P}_1$ is stabilized by some compensator $\mathbf{C}$ on $\mathcal{L}_2(-\infty,\infty)$, then it turns out that the graph of $\mathbf{P}_1$ must be closed. (This is a standard argument which proceeds as follows. Take a Cauchy sequence in the graph of $\mathbf{P}_1$ and set this equal to $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$. Since by assumption the feedback equations have unique solutions they must be given by: $u_1 = d_1$, $y_1 = d_2$, $u_2 = 0$, $y_2 = 0$. Now take the limit of the Cauchy sequence at the external inputs. Since the closed-loop operators are bounded, this pair of signals must also appear at u_1, y_1.) Thus, the graph of $\mathbf{P}_1$ must contain the closure of (2). But the closure of the infinite union (2) equals the graph of $\mathbf{P}_2$! Moreover, since $\mathbf{P}_2$ is an anticausal operator, then there are $\mathcal{L}_2(-\infty,\infty)$ input-ouput pairs which satisfy the convolution representation for $\mathbf{P}_2$ but not for $\mathbf{P}_1$. Such a pair is: $u(t) = e^{-t}$ $(t \geq 0)$, 0 $(t < 0)$ and $y(t) = -e^{-|t|}/2$, which equals

$$\begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{(s-1)(s+1)} \end{pmatrix} \tag{3}$$

in the transform domain. Thus, it seems incorrect to close the graph of $\mathbf{P}_1$ and use the ordinary definition of input-output stability. On the other hand, stabilizability of $\mathbf{P}_1$ would require that any pair of possible disturbance signals d_1, d_2 could act on the feedback system in Figure 1, and produce a bounded response. However, if a disturbance

$$\begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \end{pmatrix} = \begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix},$$

with $\hat{u}, \hat{y}$ as in (3), is allowed to act on the input ports in Figure 1, then there exists no solution which is consistent with the feedback equations and the integral representation of $\mathbf{P}_1$.

Of the following possible remedies none seems to be satisfactory:

1. To consider $\mathbf{P}_1$ to be non-stabilizable on $\mathcal{L}_2(-\infty,\infty)$,

2. To seek an alternative definition of closed-loop stability for the $\mathcal{L}_2(-\infty,\infty)$ case which would agree with the common belief that $\mathbf{P}_1$ is stabilized by proportional negative feedback of gain greater than one,

3. To identify the systems $\mathbf{P}_1, \mathbf{P}_2$.

It should be noted, that the use of extended spaces does not improve the situation, since the difficulty is to determine the correct behaviour for signals in $\mathcal{L}_2(-\infty,\infty)$.

Option (1) is a correct conclusion based on existing definitions. However, this would mean that the doubly-infinite set-up is of limited interest for control purposes, since systems which in the usual sense are "open loop unstable" would have to be excluded. Option (2) does not seem to provide any satisfactory alternatives. It is, of course, possible to restrict attention to the subspace of $\mathcal{L}_2$ consisting of functions which have support on some interval $[T, \infty)$ for some arbitrary finite T. This would treat the graph of $\mathbf{P}_1$ precisely as (2) and would work with driving signals in $\bigcup_{T \geq 0} e^{sT} \mathcal{H}_2$, which is not a closed subspace of $\mathcal{L}_2^2(-j\infty, j\infty)$. However, this appears to be a rather cosmetic solution which more or less amounts to working on the half-line. A more natural avenue would be to consider the actual $\mathcal{L}_2(-\infty, \infty)$ graph of the convolution operator $\mathbf{P}_1$. Similar reasoning to the above shows that this is the same as the graph of $\mathbf{P}_2$, but with the restriction that the inputs satisfy:

$$\int_{-\infty}^{\infty} e^{-\tau} u(\tau) d\tau = 0.$$

Again, this means that the graph $\mathbf{P}_1$ is not a closed subspace of $\mathcal{L}_2^2(-\infty, \infty)$ and the problem is then to find a suitable subspace for the external driving signals. The option of trying to work on some subspace of $\mathcal{L}_2(-\infty, \infty)$ with signals which "decay sufficiently fast" towards minus infinity again does not seem to offer a satisfying resolution. Option (3), although unsatisfactory, is perhaps not as outrageous as would first appear. If we consider the system represented by the differential equation

$$\dot{y} = y + u$$

we can reproduce the trajectories of $\mathbf{P}_1$ by solving this equation forwards in time and those of $\mathbf{P}_2$ by solving it backwards. This more or less amounts to abandonning any notion of causality (compare with the need to consider the anticausal trajectory (3) as a valid input-output pair for $\mathbf{P}_1$). However, this is not a natural option if the direction of time is well-defined (which is a basic assumption if we consider questions of control.)

Conclusion

The purpose of this note has been to point out certain features of input-output *control* theory on the doubly infinite time axis which appear intrinsically unsatisfactory. The difficulties are *not* of a mathematical nature—a consistent picture is obtained with a variety of definitions. The problem lies in trying to escape from conclusions which limit the engineering relevance of the theory, e.g. among the causal systems, only the stable ones are stabilizable. If the definitions lead to such a conclusion then it would not seem worthwhile to develop an elaborate theory of stabilization in that context. In contrast we remark that input-output *systems* theory on the doubly infinite time axis does have many important uses e.g. in discussions of fundamental limitations in filtering imposed by causality conditions, or in system approximation using Hankel operators.

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