Metric Uncertainty and Nonlinear Feedback Stabilization

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Dedicated to George Zames on the occasion of his 60th birthday

Abstract. In this paper we define a generalization of the gap metric for nonlinear systems. We prove two results which specialize to known results for linear systems. First, we show that a feedback system remains stable if the distance between the plant and its perturbation is less than the inverse of the norm of a certain parallel projection operator. Second, we show that every "continuously robust" metric on nonlinear systems is equivalent to the metric introduced in this paper.

For anyone working on the foundations of control theory it is a privilege to pay tribute to George Zames, who has contributed such fertile and fundamental ideas to the field. Over the past 35 years his work has had the most profound influence on the development of control science as well as being an inspiration to countless researchers. To honour his work we seek, in this paper, to synthesize two of his major ideas: the small gain theorem[1] in nonlinear feedback and gap metric uncertainty[2]. To date, these two important tools have experienced largely separate evolutions.

The setting for an input-output/operator view of feedback control is provided by the block diagram of Fig. 1. The plant $\mathbf{P}$ and compensator $\mathbf{C}$ are operators, possibly nonlinear, unbounded, with $\mathbf{P} : \mathcal{D}_{\mathrm{P}} \to \mathcal{Y}$ (where $\mathcal{D}_{\mathrm{P}} \subseteq \mathcal{U}$ is the domain of $\mathbf{P}$), and $\mathbf{C} : \mathcal{D}_{\mathrm{C}} \to \mathcal{U}$ (where $\mathcal{D}_{\mathrm{C}} \subseteq \mathcal{Y}$ is the domain of $\mathbf{C}$). The signals u_i ($i \in \{0,1,2\}$) belong to a Banach space $\mathcal{U}$ and the signals y_i ($i \in \{0,1,2\}$) belong to a Banach space $\mathcal{Y}$. Typical choices for $\mathcal{U}$ and $\mathcal{Y}$ are the $\mathcal{L}_p$ ($p \in \{1,2,\infty\}$) spaces of vector-valued functions on the continuous or discrete semi-infinite time axis.

The problem of feedback stabilization is to ensure that, for any $(u_0, y_0) \in \mathcal{U} \times \mathcal{Y}$, there exist unique signals $u_1, u_2 \in \mathcal{U}$ and $y_1, y_2 \in \mathcal{Y}$ such that the following feedback equations hold:

$$
\begin{aligned}
u_0 &= u_1 + u_2, \\
y_0 &= y_1 + y_2, \\
y_1 &= \mathbf{P} u_1, \\
u_2 &= \mathbf{C} y_2.
\end{aligned}
$$

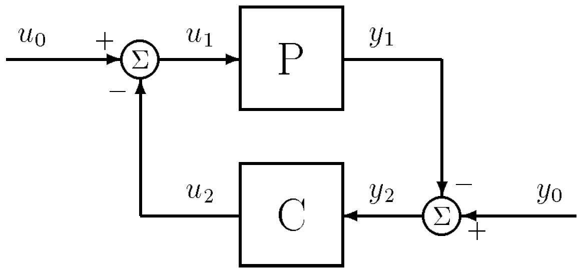

Figure 1: Standard feedback configuration.

An additional requirement is that the operator

$$\mathbf{H}_{\mathrm{P,C}} : \mathcal{W} \to \mathcal{W} \times \mathcal{W},$$

where $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$, defined via the feedback equations by

$$\mathbf{H}_{\mathrm{P,C}} : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \to \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right),$$

is bounded. In this paper we will use the incremental gain $||\cdot||_\Delta$ for operators $\mathbf{A} : \mathcal{X} \to \mathcal{Z}$,

$$||\mathbf{A}||_\Delta := \sup_{x_1, x_2 \in \mathcal{X},\ x_1 \neq x_2} \frac{||\mathbf{A}x_1 - \mathbf{A}x_2||}{||x_1 - x_2||},$$

where $||\cdot||$ denotes the norm in the Banach spaces $\mathcal{X}, \mathcal{Z}$. Thus we say that the feedback configuration of Fig. 1, denoted by $[\mathbf{P}, \mathbf{C}]$, is *stable* if $\mathbf{H}_{\mathrm{P,C}}$ is well-defined and $||\mathbf{H}_{\mathrm{P,C}}||_\Delta < \infty$. Henceforth, without loss of generality, we will assume that the zero elements in $\mathcal{U}$ and $\mathcal{Y}$ are included in the domains of the plant and compensator respectively, and that $\mathbf{P}0 = 0$, $\mathbf{C}0 = 0$.[3] Note that, under these conditions, uniqueness of solution of the feedback equations implies $\mathbf{H}_{\mathrm{P,C}}\, 0 = 0$.

Consider the *graph* of $\mathbf{P}$

$$\mathcal{G}_{\mathrm{P}} := \left\{ \begin{pmatrix} u \\ \mathbf{P}u \end{pmatrix} \ :\ u \in \mathcal{U},\ \mathbf{P}u \in \mathcal{Y} \right\} \subset \mathcal{W}$$

and the *inverse graph* of $\mathbf{C}$

$$\mathcal{G}'_{\mathrm{C}} := \left\{ \begin{pmatrix} \mathbf{C}y \\ y \end{pmatrix} \ :\ y \in \mathcal{Y},\ \mathbf{C}y \in \mathcal{U} \right\} \subset \mathcal{W}.$$

Let $\mathcal{M} := \mathcal{G}_{\mathrm{P}}$, $\mathcal{N} := \mathcal{G}'_{\mathrm{C}}$, and define the operator

$$\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}} \ : \ \mathcal{M} \times \mathcal{N} \to \mathcal{W} : (m, n) \to m + n.$$

The stability of the feedback system in Fig. 1 is equivalent to $\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}$ having an inverse defined on the whole of $\mathcal{W}$ which is bounded. In fact, if $\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}$ has a bounded inverse, the inverse is precisely $\mathbf{H}_{\mathrm{P,C}}$.[4] It can be shown that a necessary condition for $[\mathbf{P}, \mathbf{C}]$ to be stable is that $\mathcal{M}$, $\mathcal{N}$ are closed subsets of $\mathcal{W}$.[5]

Let $\mathcal{X}$, $\mathcal{Y}$ be closed subsets of a Banach space $\mathcal{W}$. We define[6]

$$\begin{aligned}
\vec{\delta}(\mathcal{X}, \mathcal{Y}) \ &:= \ \begin{cases} \inf \{ \|\mathbf{\Phi} - \mathbf{I}\|_{\Delta} : \mathbf{\Phi} \text{ is invertible on } \mathcal{W}, \\ \qquad\qquad\qquad \text{with } \mathbf{\Phi} 0 = 0, \text{ and } \mathbf{\Phi}\mathcal{X} = \mathcal{Y} \}, \\ \\ \infty \text{ if no such operator } \mathbf{\Phi} \text{ exists}, \end{cases} \\
\delta(\mathcal{X}, \mathcal{Y}) \ &:= \ \max \left\{ \vec{\delta}(\mathcal{X}, \mathcal{Y}), \vec{\delta}(\mathcal{Y}, \mathcal{X}) \right\},
\end{aligned}$$

and

$$d(\mathcal{X}, \mathcal{Y}) \ := \ \log(1 + \delta(\mathcal{X}, \mathcal{Y})).$$

Then $d(\cdot, \cdot)$ is a metric on the closed subsets of $\mathcal{W}$.[7] In general $\delta(\cdot, \cdot)$ does not satisfy the triangular inequality. Yet both $d(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ define the same topology. Moreover, $\delta(\cdot, \cdot)$ is a natural generalization of the gap (see Note 7). Hence, below, we will express our conclusions in terms of $\delta(\cdot, \cdot)$.

We now present a basic lemma which will be used in the proof of our two main results to follow. The lemma can also be used to prove the small gain theorem (see Note 1).

Lemma. Let $\mathbf{A} : \mathcal{X} \to \mathcal{X}$ where $\mathcal{X}$ is a Banach space and suppose $\|\mathbf{A}\|_{\Delta} \leq \zeta < 1$. Then

$$(\mathbf{I} + \mathbf{A})x = y \tag{1}$$

has a unique solution for all $y \in \mathcal{X}$. Furthermore,

$$\|(\mathbf{I} + \mathbf{A})^{-1} - \mathbf{I}\|_{\Delta} \leq \frac{\zeta}{1 - \zeta}.$$

Proof. The equation (1) can be rearranged as

$$x = y - \mathbf{A}x =: \mathbf{B}x.$$

Since

$$\frac{\|\mathbf{B}x_1 - \mathbf{B}x_2\|}{\|x_1 - x_2\|} = \frac{\|\mathbf{A}x_1 - \mathbf{A}x_2\|}{\|x_1 - x_2\|} \leq \zeta,$$

the contraction mapping theorem[8] applies to show that $\mathbf{B}$ has a unique fixed point. Since y is arbitrary this proves the first part of the lemma. Next note that

$$\begin{aligned}
\|(\mathbf{I}+\mathbf{A})x_1 - (\mathbf{I}+\mathbf{A})x_2\| \geq \|x_1 - x_2\| - \|\mathbf{A}x_1 - \mathbf{A}x_2\| \\
\geq \quad (\frac{1}{\zeta} - 1)\|\mathbf{A}x_1 - \mathbf{A}x_2\| \\
= \quad (\frac{1}{\zeta} - 1)\|\left(\mathbf{I} - (\mathbf{I}+\mathbf{A})\right)x_1 - \left(\mathbf{I} - (\mathbf{I}+\mathbf{A})\right)x_2\|
\end{aligned}$$

which implies that

$$\frac{\|\left((\mathbf{I}+\mathbf{A})^{-1} - \mathbf{I}\right)y_1 - \left((\mathbf{I}+\mathbf{A})^{-1} - \mathbf{I}\right)y_2\|}{\|y_1 - y_2\|} \leq \frac{\zeta}{1-\zeta}. \qquad \square$$

Let $\mathbf{\Pi}_i : \mathcal{W}\times\mathcal{W} \to \mathcal{W}$ denote the natural projection onto the ith component $(i = 1, 2)$ of $\mathcal{W}\times\mathcal{W}$. If $[\mathbf{P}, \mathbf{C}]$ is stable define $\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}} := \mathbf{\Pi}_1\mathbf{H}_{\mathrm{P,C}}$. Then

$$\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}} : \mathcal{W} \to \mathcal{M}$$

is a nonlinear idempotent (parallel projection) operator onto the graph of $\mathbf{P}$.[9] The theorem below gives a basic robustness result for uncertainties measured by $\delta(\cdot,\cdot)$ which generalizes a standard result from the linear case.[10]

Theorem 1. Let $[\mathbf{P}, \mathbf{C}]$ be stable. If

$$\delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1}) < \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{\Delta}^{-1},$$

then $[\mathbf{P}_1, \mathbf{C}]$ is stable.

Proof. Since $\delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1}) < \infty$ there exists a bounded invertible transformation $\mathbf{\Phi} : \mathcal{W} \to \mathcal{W}$ such that

$$\|\mathbf{\Phi} - \mathbf{I}\|_{\Delta} < \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{\Delta}^{-1}, \tag{2}$$

and $\mathbf{\Phi}\mathcal{M} = \mathcal{M}_1$ (where $\mathcal{M}_1 = \mathcal{G}_{\mathrm{P}_1}$). Define

$$\hat{\mathbf{\Phi}} : \mathcal{M}\times\mathcal{N} \to \mathcal{M}_1\times\mathcal{N}$$

by $\hat{\mathbf{\Phi}}(m, n) = (\mathbf{\Phi}m, n)$ and note that $\hat{\mathbf{\Phi}}$ is invertible. Consider

$$\mathbf{\Sigma}_{\mathcal{M}_1,\mathcal{N}}\hat{\mathbf{\Phi}}\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} : \mathcal{W} \to \mathcal{W}.$$

It follows that

$$\begin{aligned}
\mathbf{\Sigma}_{\mathcal{M}_1,\mathcal{N}}\hat{\mathbf{\Phi}}\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} &= (\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}} + \mathbf{\Sigma}_{\mathcal{M}_1,\mathcal{N}}\hat{\mathbf{\Phi}} - \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}})\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} \\
&= \mathbf{I} + (\mathbf{\Sigma}_{\mathcal{M}_1,\mathcal{N}}\hat{\mathbf{\Phi}} - \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}})\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} \\
&= \mathbf{I} + (\mathbf{\Phi} - \mathbf{I})\mathbf{\Pi}_1\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} \\
&= \mathbf{I} + (\mathbf{\Phi} - \mathbf{I})\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}.
\end{aligned} \tag{3}$$

The third step above follows from

$$\begin{aligned}(\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{N}}\hat{\boldsymbol{\Phi}} - \boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}})(m,n) &= \boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{N}}(\boldsymbol{\Phi}m,n) - \boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}(m,n) \\ &= \boldsymbol{\Phi}m - m \\ &= (\boldsymbol{\Phi} - \mathbf{I})m.\end{aligned}$$

From (2) we conclude that

$$\|(\boldsymbol{\Phi} - \mathbf{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|_\Delta < 1.$$

Hence, from (3) and the lemma, we conclude that $\mathbf{I} + (\boldsymbol{\Phi} - \mathbf{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ is invertible. Therefore, $\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{N}}\hat{\boldsymbol{\Phi}}\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}$ is invertible, and so is $\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{N}}$. Consequently, $[\mathbf{P}_1, \mathbf{C}]$ is stable. □

The statement of Theorem 1 can be strengthened by replacing $\delta(\cdot,\cdot)$ by $\vec{\delta}(\cdot,\cdot)$, with the same proof being valid. A similar result to Theorem 1 can be proved for uncertainty in $\mathbf{C}$, in which case the upper bound is replaced by $\|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\|_\Delta^{-1}$.[11] The result can be further extended to give guarantees on combined plant-controller uncertainty.[12]

Our second main result is a generalization to the nonlinear case of [21, Theorem 1] which asserts that open-loop uncertainties which correspond to small closed-loop errors are precisely those that are small in the gap metric. The result shows, in the language of [21], that every "continuously robust" metric on nonlinear systems is equivalent to $\delta(\cdot,\cdot)$.[13]

Theorem 2. Let $[\mathbf{P}, \mathbf{C}]$ be stable and consider a sequence of plants $\mathbf{P}_i$ for $i = 1, 2, \ldots$. Then the following statements are equivalent:

(a) $[\mathbf{P}_i, \mathbf{C}]$ is stable for sufficiently large i and $\|\mathbf{H}_{\mathrm{P,C}} - \mathbf{H}_{\mathrm{P}_i,\mathrm{C}}\|_\Delta \to 0$,

(b) $\delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_i}) \to 0$.

Proof. $(a) \Rightarrow (b)$. Let $\mathcal{M}_i := \mathcal{G}_{\mathrm{P}_i}$ and define

$$\boldsymbol{\Phi}_i := \mathbf{I}_{\mathcal{W}} - \boldsymbol{\Pi}_1(\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \boldsymbol{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}).$$

Since $\|\mathbf{H}_{\mathrm{P,C}} - \mathbf{H}_{\mathrm{P}_i,\mathrm{C}}\|_\Delta \to 0$, it follows that $\|\boldsymbol{\Pi}_1(\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \boldsymbol{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1})\|_\Delta \to 0$. Using the lemma we conclude that there exists an integer i_0 so that $\boldsymbol{\Phi}_i$ is invertible for all $i \geq i_0$. Since the graph of any system we consider contains the origin, it follows that $\boldsymbol{\Phi}_i 0 = 0$ for all i. In order to complete the proof we show that, for $i \geq i_0$, $\boldsymbol{\Phi}_i\mathcal{M} = \mathcal{M}_i$. Let $m \in \mathcal{M}$. Then

$$\begin{aligned}\boldsymbol{\Phi}_i m &= m - \boldsymbol{\Pi}_1\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}m + \boldsymbol{\Pi}_1\boldsymbol{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}m \\ &= \boldsymbol{\Pi}_1\boldsymbol{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}m \in \mathcal{M}_i.\end{aligned}$$

Let now $m_i \in \mathcal{M}_i$ and $x = \boldsymbol{\Phi}_i^{-1}m_i$. Writing

$$m_i = x - \boldsymbol{\Pi}_1\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}x + \boldsymbol{\Pi}_1\boldsymbol{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}x$$

where $(x - \mathbf{\Pi}_1 \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} x) \in \mathcal{N}$ and $\mathbf{\Pi}_1 \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} x \in \mathcal{M}_i$ we see that $\mathbf{\Pi}_1 \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} x = m_i$ and $x - \mathbf{\Pi}_1 \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} x = 0$. Therefore $x \in \mathcal{M}$, which means that $\mathbf{\Phi}_i \mathcal{M} = \mathcal{M}_i$. At the same time $\|\mathbf{\Phi}_i - \mathbf{I}\|_\Delta \to 0$. Similarly, if

$$\mathbf{\Psi}_i := \mathbf{I}_{\mathcal{W}} + \mathbf{\Pi}_1 (\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}),$$

it follows that there exists an integer i_0 so that $\mathbf{\Psi}_i$ is invertible for all $i \geq i_0$, $\mathbf{\Psi}_i \mathcal{M}_i = \mathcal{M}$, and $\|\mathbf{\Psi}_i - \mathbf{I}\|_\Delta \to 0$. (An alternative approach would be to use $\mathbf{\Psi}_i = \mathbf{\Phi}_i^{-1}$ instead. It can be shown using the lemma that

$$\|\mathbf{\Psi}_i - \mathbf{I}\|_\Delta \leq \frac{\|\mathbf{\Phi}_i - \mathbf{I}\|_\Delta}{1 - \|\mathbf{\Phi}_i - \mathbf{I}\|_\Delta}.$$

Hence, $\mathbf{\Psi}_i \mathcal{M}_i = \mathcal{M}$ and $\|\mathbf{\Psi}_i - \mathbf{I}\|_\Delta \to 0$ as well.) Therefore, $\delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_i}) \to 0$.

$(b) \Rightarrow (a)$. Suppose $\delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_i}) \to 0$. Then there exists $\mathbf{\Phi}_i$ invertible so that $\mathbf{\Phi}_i \mathcal{M} = \mathcal{M}_i$ and $\|\mathbf{\Phi}_i - \mathbf{I}\|_\Delta \to 0$. Using the notation of the proof of Theorem 1 we see, from (3) and the lemma, that $\mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}} \hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}$ is invertible for sufficiently large i and

$$\|\mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}} \hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{I}\|_\Delta \to 0$$

as $i \to \infty$. Hence $\mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}$ is invertible for sufficiently large i and $\|\mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}\|_\Delta$ is uniformly bounded. Now observe that

$$\begin{aligned}
\hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} &= \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}} \left(\hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} \right) \\
&= \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} \left(\mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}} \hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{I} \right)
\end{aligned}$$

since $\mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}$ is linear. Thus

$$\|\hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}\|_\Delta \to 0$$

as $i \to \infty$. From the identity

$$\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1} = \left(\mathbf{I} - \hat{\mathbf{\Phi}}_i \right) \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} + \hat{\mathbf{\Phi}}_i \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}$$

we see that $\|\mathbf{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} - \mathbf{\Sigma}_{\mathcal{M}_i,\mathcal{N}}^{-1}\|_\Delta \to 0$. □

Notes

1. The small gain theorem was introduced as an analysis tool for studying the stability of interconnected dynamical systems (cf. [19], [12], [2]). The

theorem applies to the feedback configuration of Fig. 1 under the assumption that **P** and **C** are stable. The precise statement is: if $\|\mathbf{P}\|_\Delta . \|\mathbf{C}\|_\Delta < 1$ then the feedback system of Fig. 1. is stable. This result, together with bounds for $\|\mathbf{H}_{\mathrm{P,C}}\|_\Delta$ (cf. [19], [2]) can be derived from the basic lemma stated in the text.

2. The gap metric was introduced into the control literature by G. Zames and A.K. El-Sakkary [21] as an analysis tool for studying perturbations of dynamical systems, and for assessing the robustness of stability of feedback interconnections. The metric defines a distance between (possibly unstable) systems by measuring the aperture between their respective graphs. The gap metric was originally used in functional analysis by B. Sz.-Nagy and by M.G. Krein and M.A. Krasnosel'skii for studying perturbations of Hilbert space operators (see [9] and the references therein).

3. If **P** and **C** do not satisfy these assumptions, biases can be introduced as suggested in [19]. Specifically, take any $b_1 \in \mathcal{D}_{\mathrm{P}}$ and define the operator $\tilde{\mathbf{P}}$ with domain $\mathcal{D}_{\tilde{\mathrm{P}}} = \mathcal{D}_{\mathrm{P}} - b_1$ by: $\tilde{\mathbf{P}}u = \mathbf{P}(u + b_1) - \mathbf{P}b_1$. We can define an operator $\tilde{\mathbf{C}}$ similarly. It can now be seen that the incremental gains of $\mathbf{H}_{\mathrm{P,C}}$ and $\mathbf{H}_{\tilde{\mathrm{P}},\tilde{\mathrm{C}}}$ are the same.

4. If $[\mathbf{P}, \mathbf{C}]$ is stable then $\mathbf{H}_{\mathrm{P,C}}\mathcal{W} \subseteq \mathcal{M} \times \mathcal{N}$. Furthermore, given any pair

$$\left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) \in \mathcal{M} \times \mathcal{N},$$

it is easy to see that for $u_0 = u_1 + u_2$ and $y_0 = y_1 + y_2$ the feedback equations are satisfied. Hence

$$\left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) = \mathbf{H}_{\mathrm{P,C}} \begin{pmatrix} u_1 + u_2 \\ y_1 + y_2 \end{pmatrix}.$$

Therefore, $\mathbf{H}_{\mathrm{P,C}}$ sets up a bijective correspondence between $\mathcal{W}$ and $\mathcal{M} \times \mathcal{N}$ such that

$$\begin{aligned} \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}} \circ \mathbf{H}_{\mathrm{P,C}} &= \mathbf{I}_{\mathcal{W}}, \\ \mathbf{H}_{\mathrm{P,C}} \circ \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}} &= \mathbf{I}_{\mathcal{M}\times\mathcal{N}}. \end{aligned}$$

The two graphs $\mathcal{M}, \mathcal{N}$ are said to *coordinatize* $\mathcal{W}$ (see [4], [6]). This property provides a characterization of stability of a feedback system. Other contributions on the significance of graph representations and on the complementarity of graphs as a condition for nonlinear feedback stability include [20], [11], [8], [16], [14].

5. The proof given in [5] applies directly to the nonlinear case. Specifically, take a Cauchy sequence in $\mathcal{M}$ and set $\begin{pmatrix} u_0 \\ y_0 \end{pmatrix}$ equal to an element of this sequence. Observe that $u_1 = u_0$ and $y_1 = y_0$ is the solution of the feedback equations. Now take limits and note that $\mathcal{M}$ must contain the limit point since $\mathcal{W}$ is complete and $\mathbf{H}_{\mathrm{P,C}}$ is continuous.

6. The definition of $\vec{\delta}(\cdot,\cdot)$ can be relaxed slightly without affecting the metric property of $d(\cdot,\cdot)$, or the subsequent theorems, in that $\mathbf{\Phi}$ need not be defined on the whole of $\mathcal{W}$. It is sufficient to define: $\vec{\delta}(\mathcal{X},\mathcal{Y}) := \inf\{\|(\mathbf{\Phi}-\mathbf{I})|_{\mathcal{X}}\|_\Delta : \mathbf{\Phi}$ is an invertible map from $\mathcal{X}$ onto $\mathcal{Y}$ with $\mathbf{\Phi}0 = 0\}$, and ∞ if no such operator $\mathbf{\Phi}$ exists.

7. This metric is a slight modification of a metric introduced in [10, page 562] for subspaces of a Banach space. The definition of the metric in [10] differs from the one in this paper in that ∞ is replaced by 1. The value ∞ is necessary in our case since we seek to measure distances between closed subsets of a Banach space which may not necessarily be subspaces. When the underlying space is a Hilbert space, $\delta(\cdot,\cdot)$ coincides with the usual gap metric [1, Corollary 3.3]. We now show that $d(\cdot,\cdot)$ is a metric.

   Clearly, $d(\mathcal{X},\mathcal{Y}) \geq 0$. If $d(\mathcal{X},\mathcal{Y}) = 0$ then $\delta(\mathcal{X},\mathcal{Y}) = 0$. It follows that, given any $\epsilon > 0$, there exists an invertible mapping $\mathbf{\Phi}_\epsilon$ such that, $\mathbf{\Phi}_\epsilon 0 = 0$, $\mathbf{\Phi}_\epsilon \mathcal{X} = \mathcal{Y}$, and $\|\mathbf{\Phi}_\epsilon - \mathbf{I}\|_\Delta < \epsilon < 1$. For any given $y \in \mathcal{Y}$ there exists a family $x_\epsilon \in \mathcal{X}$ such that $\mathbf{\Phi}_\epsilon x_\epsilon = y$. Moreover,

$$\begin{aligned} \|y - x_\epsilon\| &= \|\mathbf{\Phi}_\epsilon x_\epsilon - x_\epsilon\| = \|(\mathbf{\Phi}_\epsilon - \mathbf{I})x_\epsilon\| \\ &< \epsilon\|x_\epsilon\| < \frac{\epsilon}{1-\epsilon}\|y\|, \end{aligned}$$

   where the first inequality follows from the fact that the incremental gain bounds the norm. Hence we can find a sequence $x_\epsilon \in \mathcal{X}$ such that $x_\epsilon \to y$. Since $\mathcal{X}$ is closed then $y \in \mathcal{X}$ which means that $\mathcal{X} \subseteq \mathcal{Y}$. The reverse inclusion is proved similarly, so $\mathcal{X} = \mathcal{Y}$. Next we note that $d(\mathcal{X},\mathcal{Y}) = d(\mathcal{Y},\mathcal{X})$. Finally we show that the triangular inequality holds:

$$d(\mathcal{X},\mathcal{Z}) \leq d(\mathcal{X},\mathcal{Y}) + d(\mathcal{Y},\mathcal{Z})$$

   where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are all closed subsets of $\mathcal{W}$. If either of $d(\mathcal{X},\mathcal{Y})$, $d(\mathcal{Y},\mathcal{Z})$ is ∞, then the statement is obvious. So assume that both $d(\mathcal{X},\mathcal{Y})$, $d(\mathcal{Y},\mathcal{Z})$ are less than ∞ and, consequently, that there exist invertible mappings $\mathbf{\Phi}_i$ $(i = 1, 2)$ for which $\mathbf{\Phi}_1\mathcal{X} = \mathcal{Y}$ and $\mathbf{\Phi}_2\mathcal{Y} = \mathcal{Z}$. Then $\mathbf{\Phi} = \mathbf{\Phi}_2\mathbf{\Phi}_1$ is invertible and maps $\mathcal{X}$ onto $\mathcal{Z}$. From the identity

$$\mathbf{\Phi}_2\mathbf{\Phi}_1 - \mathbf{I} = (\mathbf{\Phi}_2 - \mathbf{I})\mathbf{\Phi}_1 + \mathbf{\Phi}_1 - \mathbf{I}$$

   we obtain

$$\begin{aligned} \|\mathbf{\Phi}_2\mathbf{\Phi}_1 - \mathbf{I}\|_\Delta &\leq \|\mathbf{\Phi}_2 - \mathbf{I}\|_\Delta \|\mathbf{\Phi}_1\|_\Delta + \|\mathbf{\Phi}_1 - \mathbf{I}\|_\Delta \\ &\leq \|\mathbf{\Phi}_2 - \mathbf{I}\|_\Delta(\|\mathbf{\Phi}_1 - \mathbf{I}\|_\Delta + 1) + \|\mathbf{\Phi}_1 - \mathbf{I}\|_\Delta. \end{aligned}$$

   This shows that

$$\log(1 + \|\mathbf{\Phi}_2\mathbf{\Phi}_1 - \mathbf{I}\|_\Delta) \leq \log(1 + \|\mathbf{\Phi}_1 - \mathbf{I}\|_\Delta) + \log(1 + \|\mathbf{\Phi}_2 - \mathbf{I}\|_\Delta)$$

   and completes the proof.

We remark that if the requirement $\mathbf{\Phi} 0 = 0$ is removed in the definition of $\vec{\delta}(\cdot,\cdot)$ then d becomes a *pseudo-metric*, because in this case $d(\mathcal{X}, \mathcal{X}+w) = 0$ for any $w \in \mathcal{W}$. That is, the δ-distance between graphs of systems differing only by a constant bias term, is zero. However, Theorem 1 is still valid in this case.

8. The Contraction Mapping Theorem of S. Banach has the following statement. Let $\mathcal{X}$ be a complete metric space with metric $d(\cdot,\cdot)$ and let $f : \mathcal{X} \to \mathcal{X}$ be a contraction mapping, i.e., $d(f(x_1), f(x_2)) \leq \zeta d(x_1, x_2)$ for some constant $\zeta < 1$ and all $x_1, x_2 \in \mathcal{X}$. Then there is a unique fixed point, i.e., there exists a unique $x \in \mathcal{X}$ such that $f(x) = x$ (see [15, p. 130]).

9. In [4] the notion of a (nonlinear) parallel projection operator was defined and studied. It was shown that there is an intimate connection between the existence of such operators and the stability of feedback systems.

10. For $\mathbf{P}$, $\mathbf{P}_1$ and $\mathbf{C}$ linear, Theorem 1 is the sufficiency part of [7, Theorem 5] and [5, Theorem 3] which asserts that the inverse of the norm of the parallel projection is the stability margin for gap metric uncertainty. Thus, the bound of Theorem 1 is tight for linear systems. The proofs of necessity given in [7], [5] do not generalize in an obvious way to the nonlinear case, and it is likely that special assumptions on $\mathbf{P}$ and $\mathbf{C}$ would be needed to show tightness, perhaps along the lines of [13].

11. The two parallel projections are related by the fact that

$$\begin{aligned} \mathbf{\Pi}_{\mathcal{M}//\mathcal{N}} + \mathbf{\Pi}_{\mathcal{N}//\mathcal{M}} &= \mathbf{\Pi}_1 \circ \mathbf{H}_{\mathrm{P,C}} + \mathbf{\Pi}_2 \circ \mathbf{H}_{\mathrm{P,C}} \\ &= \mathbf{\Sigma}_{\mathcal{M},\mathcal{N}} \circ \mathbf{H}_{\mathrm{P,C}} \\ &= \mathbf{I}_{\mathcal{W}} \end{aligned}$$

so their norms differ by at most 1. In fact, $\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\| = \|\mathbf{\Pi}_{\mathcal{N}//\mathcal{M}}\|$ under very general circumstances, e.g. if $\mathcal{W}$ is a Hilbert space, $\|\cdot\|$ denotes the usual induced norm and one of $\mathcal{M}$ or $\mathcal{N}$ is a linear subspace [4].

12. To do this, the following bound can be calculated from (3) and the lemma

$$\left\| \left(\mathbf{\Sigma}_{\mathcal{M}_1,\mathcal{N}} \hat{\mathbf{\Phi}} \mathbf{\Sigma}^{-1}_{\mathcal{M},\mathcal{N}} \right)^{-1} \right\|_{\Delta} < \frac{1}{1 - \delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1}) \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{\Delta}}$$

from which an upper bound on $\|\mathbf{\Sigma}^{-1}_{\mathcal{M}_1,\mathcal{N}}\|_{\Delta}$ and thence on $\|\mathbf{\Pi}_{\mathcal{N}//\mathcal{M}_1}\|_{\Delta}$ can be found.

13. An alternative generalization of the gap for nonlinear systems was introduced in [6]. The topology in [6] is a *pseudo-metric* topology and applies to systems with differentiable graphs (modelled as infinite-dimensional manifolds over a Banach space). However, the topology in [6] is not "continuously robust" in the sense of [21]. Two systems are close in the

sense of [6] if they have similar incremental responses, albeit about possibly different input/output operating conditions. Thereby, small values for $||\mathbf{H}_{\mathrm{P},\mathrm{C}} - \mathbf{H}_{\mathrm{P}_1,\mathrm{C}}||_\Delta$ are not required for two open-loop plants $\mathbf{P}, \mathbf{P}_1$ to be close. Yet it is interesting to note that the stability of a feedback system is still a robust property with respect to the topology in [6], in spite of the fact that this topology is coarser than the one induced by $\delta(\cdot,\cdot)$, when $\delta(\cdot,\cdot)$ is specialized to systems with differentiable graphs. (A topology $\mathcal{T}$ is coarser than $\mathcal{T}_1$ if $\mathcal{T} \subset \mathcal{T}_1$ [15, p. 47].) We would like to mention that a different metric topology for nonlinear systems is proposed in [22]. Unfortunately, feedback stability is not a robust property for the topology in [22]. Finally it is worth noting that, for the case of linear systems, the question of whether the gap/graph topology ([17]) is the coarsest topology in which stability is a robust property remains open (see [18, page 44], [3]).

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