On $\mathcal{L}_2$ error bounds between systems

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Abstract

In this paper two systems described by nonlinear ordinary differential equations will be considered. They will be assumed to be operating independently and to be time-invariant, unforced, and equidimensional. Dissipativity theory is used to provide a sufficient condition for the componentwise difference that exists between their state vectors to be a finite-energy signal. The related set of storage functions will be used to provide a bound for its $\mathcal{L}_2$-norm. For a specific class of switched models, a heuristic methodology to construct such storage functions is presented. Special emphasis will be placed upon piecewise linear (PWL) models.

1 Introduction

In this paper, we will consider two continuous-time dynamical models, denoted a and b, operating independently. We can think of a as the complicated but accurate description of a real-life system and of b as a simplified version. Quantifying how different their behaviour is for all time $t \geq 0$ would shed some light on how close the models are in terms of representing the same phenomenon. This question will be addressed here for the case in which one model is given by a particular class of nonlinear ordinary differential equations and the other one is piecewise linear (PWL).

PWL models are pervasive in a wide spectrum of applications. For example, [2] and [4] present PWL circuit models of hysteresis phenomena. In [3], the response of a transmission line to a ramp input is approximated via a PWL waveform. PWL approximations of power electronics systems are used in [12] to cut down the time it takes to simulate such systems. In [9] a class of nonlinear filters is explained by resorting to the theory of PWL functions. And it is also known that computer graphics and scientific visualisation algorithms require the approximation of data by linear pieces [13]. This multifarious applicability has motivated the emergence of dedicated analysis methodologies which, as the results on stability of PWL models [8], [5], [6], and [7] exemplify, tend to disregard the relationship that exists between the model and the phenomenon it is representing by focusing entirely on the former and ignoring the latter. This situation constitutes the motivating force behind the methodology presented herein.

Built upon the well-known dissipativity theory [16], section 2 presents a sufficient condition which ensures that the component-wise difference that exists between the trajectories of models a and b is a finite energy signal; furthermore, it provides a bound for the energy of such signal. For a particular class of models, an algorithmic approach to the aforementioned condition is introduced in section 3. Its use is illustrated in the examples provided in the subsequent section. This paper concludes with a discussion on the advantages and limitations of the obtained results.

2 Preliminaries

In this section we introduce the concepts which will be the basis for subsequent developments.

Take $i = \{a, b\}$ and let $x_i \in \mathbb{X}_i = \mathbb{R}^n$ be the state vector of the dynamical model $\dot{x}_i = f_i(x_i)$. Restrict $\dot{x}_i = f_i(x_i)$ to have a unique solution or *trajectory* $x_i(\cdot) : \mathbb{R} \to \mathbb{X}_i$ for any initial condition $x_i(0) \in \mathbb{X}_i$ and denote the set of solutions as $\mathcal{B}_i$. Define the $2n$-tuple $x \in \mathbb{X}_a \times \mathbb{X}_b$ as $x := [x_a^T \; x_b^T]^T$, the function $y : \mathcal{B}_a \times \mathcal{B}_b \to \mathbb{R}^n$ as $y := x_a(t) - x_b(t)$ and the vector-valued map $f : \mathbb{X}_a \times \mathbb{X}_b \to \mathbb{R}^n \times \mathbb{R}^n$ as $f(x) := [f_a^T(x_a) \; f_b^T(x_b)]^T$.

In the preceding paragraph, two independent, unforced, continuous-time models of the same dimension are being stacked to create a new one of the form

$$\begin{aligned} \dot{x} &= f(x) \\ y &= g(x) \end{aligned} \tag{1}$$

where the evolution over time of the two constituent models is compared by means of the component-wise difference that exists between their state vectors, namely y. It is clear that $x_a(t) \to x_b(t)$ as $t \to \infty$ is only a necessary condition for y to be a finite energy signal. The following proposition provides a sufficient condition which ensures that this is indeed the case and provides an upper bound on the energy of y.

Proposition 1. *Let the* supply rate *function $w : \mathbb{R}^n \to \mathbb{R}$ be defined by $w := -y^T y$. If there exists a nonnegative* storage function $V : \mathcal{B}_a \times \mathcal{B}_b \to [0, \infty)$ *such* *that the* dissipation inequality

$$V(x(t_0)) - V(x(t_1)) \geq -\int_{t_0}^{t_1} w\, dt \tag{2}$$

is fulfilled for all $x(t_0), x(t_1) \in \mathcal{B}_a \times \mathcal{B}_b$ *and* $t_0 < t_1$*, then* $y \in \mathcal{L}_2$*, i.e.* y *is a finite energy signal. Furthermore,* $\|y\|_{\mathcal{L}_2} := \sqrt{\int_0^\infty y^T y\, dt} \leq \sqrt{V(x(0))}$.

Proof. As the storage function $V(x(t))$ is nonnegative, the inequalities

$$\begin{aligned} V(x(t_0)) &\geq V(x(t_1)) + \int_{t_0}^{t_1} y^T y \; dt \\ &\geq \int_{t_0}^{t_1} \left(x_a(t) - x_b(t)\right)^T \left(x_a(t) - x_b(t)\right) \; dt \end{aligned} \tag{3}$$

follow immediately from the definition of w and y and the fact that V is nonnegative. Letting $t_0 = 0$ and $t_1 \to \infty$ in (3) and taking the square root in both sides completes the proof. □

Although dissipativity theory is an important theoretical tool, a constructive methodology to find storage functions is not available for all kinds of models. And even for those cases which do have an associated constructive methodology, the convex set of possible storage functions (see [16]) is not necessarily a singleton. In the forthcoming section we will restrict ourselves to a particular class of models; this allows us to advance a methodology to construct storage functions as well as criteria to select one which tightens the bound for $\|y\|_{\mathcal{L}_2}$ mentioned in proposition 1.

3 Algorithmic implementation

In this section, we take model a to have a polynomial nonlinear vector field $f_a(x_a)$ and model b to be PWL. In addition, we will restrict ourselves to the space of polynomial functions when looking for a storage function V that satisfies proposition 1. This opens the door to an algorithmic approach to proposition 1 based on the SOS decomposition of multivariate polynomials [11]; an example of the attainable results can be found in section 4.1. Moreover, all propositions included in this section can be easily adapted to consider models with non-polynomial vector fields provided they are recasted according to [10]. While the details are easy to carry out and have been omitted, section 4.2 presents an example which is assumed to be derived from such a recasting process.

Let model a have a nonlinear vector field $f_a(x_a)$ and model b be PWL. Then (1) can be expressed as

$$\begin{aligned} \dot{x} &= F_l(x), \quad l \in L = \{1, \dots, N\} \\ y &= g(x) \end{aligned} \tag{4}$$

where $x \in \mathbb{X}_a \times \mathbb{X}_b$ is the continuous state, l is the discrete state, $F_l(x)$ is the vector field describing the dynamics of the l-th mode and L is the index set. This is a switched system with a polynomial vector field, and is said to be in the l-th mode at time t if $x(t) \in X_l$, where $X_l \subset \mathbb{X}_a \times \mathbb{X}_b$ is a region of the state space described by

$$X_l = \{x \in \mathbb{X}_a \times \mathbb{X}_b \mid \phi_{lh}(x) \leq 0, \text{ for } h = 1, \dots, M_{X_l}\} \tag{5}$$

for some real-valued polynomial functions ϕ_{lh}. Additionally, the state space partition $\{X_l\}$ must satisfy $\bigcup_{l \in L} X_l = \mathbb{X}_a \times \mathbb{X}_b$ and $\text{int}(X_\alpha) \bigcap \text{int}(X_\beta) = \emptyset$ for $\alpha \neq \beta$, where $\text{int}(\cdot)$ denotes the interior of a set. A switching surface between the α-th and β-th mode, i.e. a boundary between X_α and X_β, is given by

$$S_{\alpha\beta} = \{x \in \mathbb{X}_a \times \mathbb{X}_b \mid \varphi_{\alpha\beta 0}(x) = 0, \, \varphi_{\alpha\beta\gamma}(x) \leq 0, \, \gamma = 1, \dots, m_{S_{\alpha\beta}}\} \tag{6}$$

for some real-valued polynomial functions $\varphi_{\alpha\beta\gamma}$. In addition, it will be assumed that model (4) does not allow the presence of *chattering* or *sliding-modes*.

Considering model (4), we will look for several polynomial functions $V_l(x)$ (typically corresponding to the state space partition $\{X_l\}$); these polynomial functions will be concatenated in order to get the function $V(x)$ mentioned in proposition 1. As $V_l(x)$ being polynomial implies that the necessary partial derivatives exist, this amounts to asking the conditions

$$\begin{aligned} V_l(x) &\geq 0 \\ \nabla V_l(x) \cdot F_l(x) &\leq w \end{aligned} \tag{7}$$

to hold only on X_l and, if the function $V(x)$ is to be continuous, considering $V_\alpha(x) = V_\beta(x)$ for all $x \in S_{\alpha\beta}$ as well. The resulting piecewise polynomial function $V(x)$ will be defined by $V(x) = V_l(x)$ if $x \in X_l$.

Recalling that the polynomial function $\Theta(x)$ is a sum of squares (SOS) if there exist polynomials $\theta_j(x)$ such that $\Theta(x) = \sum_j (\theta_j(x))^2$, it can be seen that such condition naturally implies the non-negativeness of $\Theta(x)$. To see how we will be using this result say we want to use the S-procedure [1] to check that the condition

$$V_l(x) \geq 0 \text{ when } \phi_{lh}(x) \leq 0 \text{ for } l \in L \text{ and } h = 1, \dots, M_{X_l}$$

holds. Instead of finding *positive constant* multipliers (the standard S-procedure), we search for SOS multipliers $\hat{a}_{lh}(x)$ such that

$$V_l(x) + \sum_h \hat{a}_{lh}(x)\, \phi_{lh}(x) \text{ is a SOS for all } l \in L \tag{8}$$

Since $\hat{a}_{lh}(x) \geq 0$ and the condition (8) is satisfied, for any x such that $\phi_{lh}(x) \leq 0$ we automatically have $V_l(x) \geq 0$, so sufficiency follows. This condition is as least as powerful as the standard S-procedure, and many times it is strictly better. Besides this, what is more interesting is the case in which the monomials in the polynomial $V_l(x)$ have *unknown* coefficients, and we want to search for the values of those coefficients so that $V_l(x)$ satisfies (8). SOS programming [14], [15] easily allows this, and so it will be used to provide an algorithmic approach to proposition 1.

Proposition 2 (Global analysis). *Consider the switched model* (4)*. If there exist polynomials $V_l(x)$, $\hat{c}_{\alpha\beta}(x)$ and sums of squares $\hat{a}_{lh}(x)$, $\hat{b}_{lh}(x)$, such that*

$$\begin{gathered}
V_l(x) + \sum_{h=1}^{M_{X_l}} \hat{a}_{lh}(x)\, \phi_{lh}(x) \ \textit{is SOS for all } l \in L \\
w + \sum_{h=1}^{M_{X_l}} \hat{b}_{lh}(x)\, \phi_{lh}(x) - \nabla V_l(x) \cdot F_l(x) \ \textit{is SOS for all } l \in L \\
V_\alpha(x) + \hat{c}_{\alpha\beta}(x)\, \varphi_{\alpha\beta 0}(x) - V_\beta(x) = 0 \ \textit{for all } \alpha \neq \beta
\end{gathered} \tag{9}$$

then $V(x) = V_l(x)$ for $x \in X_l$ is a storage function in accordance with proposition 1. Hence, for any positive real constant ε, the set

$$\mathcal{X}_\varepsilon = \left\{ x \in \mathbb{X}_a \times \mathbb{X}_b \,|\, V(x) \leq \varepsilon^2 \right\} \tag{10}$$

describes the set of initial conditions $x(0) \in \mathbb{X}_a \times \mathbb{X}_b$ which guarantee that $\|y\|_{\mathcal{L}_2} \leq \varepsilon$.

Furthermore, let $z_l(x)$ be vectors of monomials in x and P_l be real symmetric matrices of appropriate dimensions such that $V_l(x) = z_l^T(x)\, P_l\, z_l(x)$ for all $l \in L$. Take γ_l for all $l \in L$ and γ_{obj} to be real numbers. Let $\|\cdot\|_2$ denote the Euclidean norm. The following SOS optimisation programme, if feasible, ensures that $V(x) = V_l(x) \leq \gamma_{obj} \|z_l(x)\|_2^2$ for all $x \in X_l$ and tightens the bound for $\|y\|_{\mathcal{L}_2}$,

$$\begin{array}{ll}
\min & \gamma_{obj} \\
s.\ t. & \gamma_l\, z_l^T(x)\, z_l(x) - V_l(x) \ \textit{is SOS for all } l \in L \\
& \gamma_{obj} - \gamma_l \geq 0 \ \textit{for all } l \in L \\
& \textit{the set of conditions (9) hold.}
\end{array} \tag{11}$$

Proof. The non-negativeness of $V_l(x)$ in X_l follows from the first condition in (9) considering the negative semi-definiteness of $\phi_{lh}(x)$ in X_l and the positive semi-definiteness of $\hat{a}_{lh}(x)$. A similar argument applied to the second condition ensures that $\nabla V_l(x) \cdot F_l(x) \leq w$ in X_l. Continuity of $V(x)$ is guaranteed by the third condition recalling that $\varphi_{\alpha\beta 0} = 0$ for all $x \in S_{\alpha\beta}$ and all $\alpha \neq \beta$. The claim about the set $\mathcal{X}_\varepsilon$ follows from proposition 1 and equation (10).

The first two constraints in (11) imply, respectively, the two inequalities $\lambda_{max}(P_l) \leq \gamma_l \leq \gamma_{obj}$, where $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue of a real symmetric matrix. These inequalities are in turn equivalent to $V_l(x) \leq \gamma_l \|z_l(x)\|_2^2 \leq \gamma_{obj} \|z_l(x)\|_2^2$, a fact which can either be seen directly from the constraints in (11) or inferred by the equality $V_l(x) = z_l^T(x)\, P_l\, z_l(x)$. It then follows that taking γ_{obj} as the minimisation objective tightens the bound for $\|y\|_{\mathcal{L}_2}$. $\square$

In principle, one could substitute (11) by the SOS programme

$$\min V_l(x(0)) \text{ subject to the set of conditions (9)}$$

for a given $x(0) \in X_l$, $l \in L$. If the programme is feasible, we obtain the smallest possible value for $V(x(0))$. As a byproduct, we also get a set $\mathcal{X}_\varepsilon$ which fulfils the characteristics mentioned in proposition 2 but for which there is no guarantee that the bound for $\|y\|_{\mathcal{L}_2}$ is reasonably tight. If a reasonably tight bound is required for a set of initial conditions $x(0) \in X_l$, $l \in L$, the fact that the minimisation of $V_l(x(0))$ would have to be done on a point-by-point basis renders the idea useless. On the other hand, as the approach presented herein via (11) is specifically tailored to consider sets of possible initial conditions rather than isolated points, it constitutes a much superior alternative.

Note that proposition 2 is suitable for the analysis of the model (4) over the entire state space $\mathbb{X}_a \times \mathbb{X}_b$. Minor modifications allows proposition 2's groundwork to be adapted to local analysis, as we shall see next.

Without loss of generality, let the origin be an equilibrium point of both models a and b. Let $\mathcal{R}$ be a bounded region of the state space $\mathbb{X}_a \times \mathbb{X}_b$ containing the origin; (i.e. $0 \subset \mathcal{R} \subset \mathbb{X}_a \times \mathbb{X}_b$); let $\mathcal{R}$ be defined by

$$\mathcal{R} = \{x \in \mathbb{X}_a \times \mathbb{X}_b \mid \psi_r(x) \leq 0, \text{ for } r = 1, \ldots, M_{\mathcal{R}}\} \tag{12}$$

for some polynomial, real-valued functions ψ_r. Let L_c be the index set of the regions X_l of the state space which are entirely contained in $\mathcal{R}$, L_n the index set of the regions X_l which are entirely outside the region $\mathcal{R}$, and L_p be the index set of the regions X_l which are partially contained in $\mathcal{R}$, viz.

$$\begin{aligned} L_c &= \{l \in L \mid X_l \cap \mathcal{R} = X_l\} \\ L_n &= \{l \in L \mid X_l \cap \mathcal{R} = \varnothing\} \\ L_p &= \{l \in L \mid X_l \cap \mathcal{R} \neq X_l,\ X_l \cap \mathcal{R} \neq \varnothing\} \end{aligned} \tag{13}$$

Denote the regions themselves by X_{l_c}, X_{l_n}, and X_{l_p}.

The region $X_{l_p} \cap \mathcal{R}$ (i.e. the part of X_{l_p} contained in $\mathcal{R}$) can be obtained by recalling the definitions of X_l and $\mathcal{R}$ given by equations (5) and (12). Let the sets $\{\phi^\star_{lh}(x)\}$ and $\{\psi^\star_{lr}(x)\}$ be respectively given by $\{\phi_{lh} \,|\, (\phi_{lh}(x) = 0) \cap \mathcal{R} \neq \varnothing\}$ and $\{\psi_r \,|\, (\psi_r(x) = 0) \cap X_l \neq \varnothing\}$ for $h = 1, \ldots, M_{X_l}$, $r = 1, \ldots, M_{\mathcal{R}}$, and a previously-determined $l \in L_p$. Define then

$$X_{l_p} \cap \mathcal{R} = \left\{ x \;\middle|\; \begin{array}{l} \phi^\star_{lh}(x) \leq 0 \text{ for } h = 1, \ldots, m_{X_l} \\ \psi^\star_{lr}(x) \leq 0 \text{ for } r = 1, \ldots, m_{\mathcal{R}} \end{array} \right\} \tag{14}$$

noticing that $m_{X_l} \leq M_{x_l}$ and $m_{\mathcal{R}} \leq M_{\mathcal{R}}$.

It is worth mentioning that determining the sets given by equations (13) and (14) is a hard problem in general. However, for some specific instances this can be readily done, as is the case in the examples presented herein.

Proposition 3 (Local analysis). *Consider the switched model* (4) *and the region* $\mathcal{R}$ *given by* (12)*. If there exist polynomials* $V_l(x)$, $\hat{g}_{\alpha\beta}(x)$ *and sums of squares* $\hat{a}_{lh}(x)$, $\hat{b}_{lh}(x)$, $\hat{c}_{lh}(x)$, $\hat{d}_{lr}(x)$, $\hat{e}_{lh}(x)$, *and* $\hat{f}_{lr}(x)$ *such that*

$$\begin{gathered} V_l(x) + \sum_{h=1}^{M_{X_l}} \hat{a}_{lh}(x)\,\phi_{lh}(x) \text{ is SOS for all } l \in L_c \\ w + \sum_{h=1}^{M_{X_l}} \hat{b}_{lh}(x)\,\phi_{lh}(x) - \nabla V_l(x) \cdot F_l(x) \text{ is SOS for all } l \in L_c \\ V_l(x) - \sum_{h=1}^{m_{X_l}} \hat{c}_{lh}(x)\,\phi^\star_{lh}(x) - \sum_{r=1}^{m_{\mathcal{R}}} \hat{d}_{lr}(x)\,\psi^\star_{lr}(x) \text{ is SOS for all } l \in L_p \\ w + \sum_{h=1}^{m_{X_l}} \hat{e}_{lh}(x)\,\phi^\star_{lh}(x) + \sum_{r=1}^{m_{\mathcal{R}}} \hat{f}_{lr}(x)\,\psi^\star_{lr}(x) - \nabla V_l(x) \cdot F_l(x) \text{ is SOS for all } l \in L_p \\ V_\alpha(x) + \hat{g}_{\alpha\beta}(x)\,\varphi_{\alpha\beta 0}(x) - V_\beta(x) = 0 \text{ for all } \alpha \neq \beta \text{ where } \alpha, \beta \in L_p \cup L_c \end{gathered} \tag{15}$$

then $V(x) = V_l(x)$ *if* $x \in X_l$ *yields a storage function* $V(x)$ *which complies with proposition 1 in the region* $\mathcal{R}$ *described by* (12)*. Hence, for any positive real constant* ε*, the set*

$$\mathcal{X}^\star_\epsilon = \left\{ x \in \mathbb{X}_a \times \mathbb{X}_b \,|\, V(x) \leq \varepsilon^2 \right\} \subseteq \mathcal{R} \tag{16}$$

describes the set of initial conditions $x(0) \in \mathcal{R}$ *which guarantee that* $\|y\|_{\mathcal{L}_2} \leq \varepsilon$.

Furthermore, let γ_l *and* γ_{obj} *be real numbers. The existence of sums of squares* $\hat{h}_{lh}(x)$, $\hat{i}_{lh}(x)$, *and* $\hat{j}_l(x)$ *such that the following SOS optimisation programme is* *feasible ensures that* $V(x) \leq \gamma_{obj}$ *for all* $x \in \mathcal{R}$ *and tightens the bound for* $\|y\|_{\mathcal{L}_2}$.

$$\begin{array}{ll} \min & \gamma_{obj} \\ s.t. & \gamma_l + \sum_{h=1}^{M_{X_l}} \hat{h}_{lh}(x)\,\phi_{lh}(x) - V_l(x) \text{ is SOS for all } l \in L_c \\ & \gamma_l + \sum_{h=1}^{m_{X_l}} \hat{\imath}_{lh}(x)\,\phi^\star_{lh}(x) + \sum_{r=1}^{m_{\mathcal{R}}} \hat{\jmath}_{lr}(x)\,\psi^\star_{lr}(x) - V_l(x) \text{ is SOS for all } l \in L_p \\ & \gamma_{obj} - \gamma_l \geq 0 \text{ for all } l \in L_c \cup L_p \\ & \text{the set of conditions (15) hold.} \end{array} \tag{17}$$

Proof. The proof follows the same lines than the one for proposition 2 and is thus omitted. □

Remark 1. *Note that the set* $\{x \in \mathcal{R} \mid V(x) \leq \gamma_{obj}\}$ *computed using proposition 3 is not necessarily contained in* $\mathcal{R}$*. Further computations are required to find the range of values of* ε *for which condition* (16) *holds.*

Ensuring that the inequalities (7) hold for all regions X_{l_c} and $X_{l_p} \cap \mathcal{R}$, together with the continuity of $V(x)$ in $\mathcal{R}$, is achieved by the SOS-based relaxations given by (15). If the region $\mathcal{R}$ were defined by

$$\mathcal{R} = \left\{ x \;\middle|\; \begin{array}{l} \psi_r(x) \leq 0, \text{ for } r = 1, \ldots, M_{\mathcal{R}} \\ x_{j_a} - x_{j_b} = 0, \text{ for } j = 1, \ldots, n \end{array} \right\}$$

rather than by (12), this would imply that the set $\mathcal{X}^\star_\varepsilon$ given by (16) would be explicitly tailored to consider only those initial conditions which fulfil $x_a(0) = x_b(0)$. Unfortunately, this requires the third condition in (15) to include terms of the form $\sum_{j=1}^{n}(x_{j_a} - x_{j_b})\,\hat{p}_{lj}(x)$, where $\hat{p}_{lj}(x)$ are polynomial functions in x. Similar terms must also be included in the fourth condition in (15) and the second constraint in (17). This increases the complexity of the SOS programme and in general introduces additional conservativeness to the aforementioned SOS relaxations, thus making the idea impractical.

4 Examples

4.1 Example 1

Let model a be given by

$$\begin{aligned} \dot{x}_{1_a} &= \frac{1}{10}x_{2_a}^3 + x_{2_a} - 0.1x_{1_a} \\ \dot{x}_{2_a} &= -x_{1_a}^3 - 0.1x_{2_a} \end{aligned} \tag{18}$$

and model b be given by

$$\begin{aligned} \dot{x_b} &= \begin{bmatrix} -0.1 & 1 \\ -1 & -0.1 \end{bmatrix} x_b \qquad && \text{for } x_{1_b} < 0 \\ \dot{x_b} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_b \qquad && \text{for } x_{1_b} \geq 0 \end{aligned} \tag{19}$$

Let the region for local analysis $\mathcal{R}$ be given by

$$\mathcal{R} = \{x \mid x_{1_a}^2 + x_{2_a}^2 - 4 \leq 0, x_{1_b}^2 + x_{2_b}^2 - 4 \leq 0\} \tag{20}$$

To illustrate the results which can be obtained with the proposed methodology, both propositions 2 and 3 will be put to use twice. In the first try, we will look for polynomials and sums of squares which contain all monomials in x of degree up to 6; in the second one, we will go up to 8. If suitable storage functions $V(x)$ can be found, the value of $\|y\|_{\mathcal{L}_2}$ will be computed for selected initial conditions which fulfil $x_a(0) = x_b(0)$ and $V(x(0)) = \varepsilon^2$ for a given ε which guarantees that $\{x \mid V(x(0)) = \varepsilon^2\} \subseteq \mathcal{R}$.

Figure 1 shows the results. As one might expect, local analysis yields better results for the region of interest $\mathcal{R}$ than the global one. It is also worth noticing how results improve as we look for polynomials and sums of squares of increasingly higher degree.

4.2 Example 2

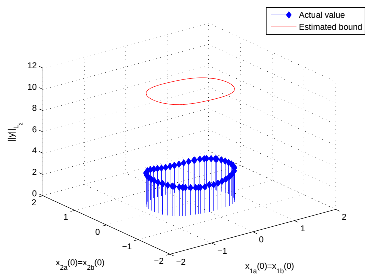
In this example, we will suppose that the PWL model given by (19) is being used to approximate a different model a and a similar analysis to that of the previous example will be carried out.

Let model a be given by

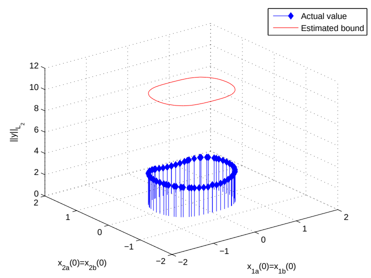
$$\begin{aligned} \dot{x}_{1_a} &= \frac{x_{2_a}^3}{x_{2_a}^2 + 1} - 0.1 x_{1_a} \\ \dot{x}_{2_a} &= -\frac{x_{1_a}^5}{x_{1_a}^4 + 5} - 0.1 x_{2_a} \end{aligned} \tag{21}$$

Model b and the region for local analysis $\mathcal{R}$ will be given by (19) and (20), respectively.

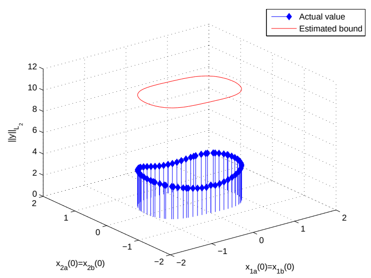
Notice that model a has a rational vector field rather than a polynomial one. In spite of that, propositions 2 and 3 still apply if all conditions which include the term $\nabla V_l(x) \cdot F_l(x)$ are multiplied by the positive definite polynomial $(x_{2_a}^2 + 1)(x_{1_a}^4 + 5)$.



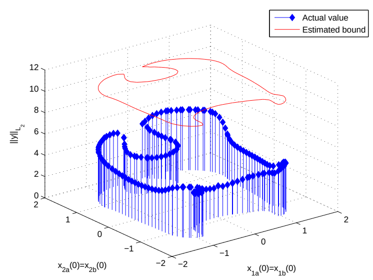
(a) Global analysis - First try.



(b) Global analysis - Second try.



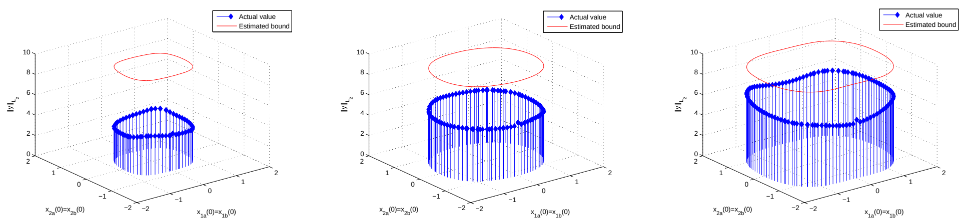
(c) Local analysis - First try.



(d) Local analysis - Second try.

Figure 1: Value of $\|y\|_{\mathcal{L}_2}$ for initial conditions which fulfils $x_a(0) = x_b(0)$ and $V(x(0)) = 109.73$. The stem plot shows the actual value while the solid line shows the bound given by $\sqrt{V(x(0))}$.

Propositions 2 and 3 will be again put to use twice. In the first try, we will look for polynomials and sums of squares which contain all monomials in x of degree up to 4; in the second one, we will go up to 6. Figure 2 shows the results. Notice that there is no plot for the first try using proposition 2 because the associated SOS programme turned out to be infeasible.



(a) Global analysis - Second try.

(b) Local analysis - First try.

(c) Local analysis - Second try.

Figure 2: Value of $||y||_{\mathcal{L}_2}$ for initial conditions which fulfill $x_a(0) = x_b(0)$ and $V(x(0)) = 86.49$. The stem plot shows the actual value while the solid line shows the bound given by $\sqrt{V(x(0))}$.

5 Concluding remarks

Proposition 1 presented a sufficient condition for the signal y - as described in section 2 - to belong to $\mathcal{L}_2$ space. For the class of systems described in section 3, propositions 2 and 3 presented a heuristic approach to build bounded-degree polynomial functions which fulfil the conditions to become the storage functions described in proposition 1.

There is a subtle but important difference between propositions 2 and 3. As $\mathcal{R}$ was defined to be a bounded region, the polynomial functions $V_l(x)$ can indeed be bounded by constants γ_l provided that x belongs to $\mathcal{R}$. This is not true for an unbounded region, so this difference explains why the sets of constraints in (11) and (17) use different approaches to tighten the bound for $\|y\|_{\mathcal{L}_2}$.

It must also be acknowledged that failing to find a storage function $V(x)$ by means of propositions 2 and/or 3 does not allow us to advance a definite conclusion regarding the finiteness of the energy of the signal y. In such a case, looking for polynomials or sums of squares of increasingly higher degrees could eventually allow us to find the required $V(x)$, as illustrated by the example presented in section 4.2.

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